Optimizing Signal Constellations

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by

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CERTIFICATE

It is certified that the work contained in the thesis entitled "*Optimizing Signal Constellations* ", by Kartik Ahuja, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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To

my Parents

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Abstract

Previous works in the area of signal constellation design to minimize symbol error rate have dealt with the problem at asymptotic SNR values. Optimal constellations which achieve minimum possible symbol error rate or bit error rate at any given SNR have not been established. In this work we come up with solutions to this problem for 1 and 2 dimensional constellation for AWGN and fading channels. Shape of optimal signal constellations varies with SNR value and this fact has interesting implications for fading channel. Depending on the channel gain, the transmitter decides the amount of power and which geometry to use to have a minimum average symbol or bit error rate, optimal solutions to this problem are arrived at. Optimal signal constellations arrived at are compared with the best ones known in literature to show the improvements. We show that necessary conditions in literature for optimality of 2 dimensional constellations at asymptotically high SNR values are inaccurate and thus, arrive at a new set of necessary conditions.

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Keywords

Signal Constellation, Symbol Error Rate, Bit Error Rate, Convex function, Hamming distance, Voronoi Diagram, KKT Conditions, Gray Coding, Coherent Detection, Interior Point Methods, Lagrangian, Moment Generating Function, Laplacian, Rayleigh Fading

Abbreviations

SER : Symbol Error Rate

BER : Bit Error Rate

AWGN: Additive White Gaussian Noise

SNR : Signal to Noise Ratio

I-Phase : In Phase

Q-Phase : Quadrature Phase

QAM : Quadrature Amplitude Modulation

PAM : Pulse Amplitude Modulation

PSK : Phase Shift Keying

Chapter 1

Introduction

Signal constellation, a set of symbols on which the transmitter encodes the message is fundamental to any form of digital communication, both wireline and wireless. 1-Dimensional Pulse Amplitude Modulated constellations were the basis of amplitude modulation. Idea of combining phase and amplitude modulation was introduced by [1],[2] and refined by [3] leading to 2 dimensional constellations as we see today. Symbol error rate which depends on the total power in the constellation and the coordinates of symbols characterizes the constellation and the performance of the communication system. Bit error rate, another important quantity characterizing the performance of communication system, depends on both coordinates of symbols and the labeling of the symbols. For a fixed power it is important to understand the constellations minimizing the symbol error rate and the work of foshcini et.al is a landmark contribution in this direction [4].

1.1 Motivation

Given a fixed average power at the transmitter, channel capacity helps one understand the maximum number of bits that can be reliably transmitted per usage of channel and coding theory has helped in realizing the goal of achieving the same. On the same lines we want to answer the question that given a fixed average power at the transmitter and a fixed signal constellation size, what is the minimum symbol/bit error rate that can be achieved and which signal constellation achieves it. Work of Foshcini et.al in the context of 2-D constellations gives the solutions to this problem but only at asymptotically high SNRs in AWGN channel and the work of Makowski et.al [5] in the context of 1-D constellation characterizes the optimal solution at any SNR but only in AWGN channel. These gaps in the understanding of best signal constellations act as the driving force of our work.

1.2 Contribution

Our contributions towards filling the gaps in the understanding of optimal constellations are,

- Necessary conditions for optimality in terms of SER/BER in AWGN channel for 2-D constellation at asymptotic SNR are derived. It turns out that characterization given by Foschini et.al for asymptotic optimality is not accurate and we show it through a counterexample. Optimal 2-D constellation at asymptotic SNR values in AWGN and rayleigh fading channel are arrived at and are shown to form a lattice of equilateral triangles.

- For any fixed SNR, we arrive at necessary conditions for optimality of 1-D constellation in terms of SER in a given fading channel. We come up with optimal solutions for both SER/BER in both AWGN and fading channels.

- For the case of 2-D constellations, optimal signal sets (8 and 16 point) in terms of SER/BER in AWGN channel which depend on SNR value are arrived at.

1.3 Organization

The organization of thesis is as follows, in the first two sections of the second chapter we present the system model and general formulation of the problem followed by literature survey on the work that has been done on some aspects of the problem. In the first section of third chapter we formulate the optimization problem for 1-D and 2-D constellations at asymptotically high SNRs in terms of SER/BER in both AWGN and fading channels. In the next section we come up with the necessary conditions for optimality in terms of SER/BER in AWGN channel for 2-D constellations followed by the section where we show the counterexample to Foschini's necessary conditions. In the second last section we show the optimal solutions for both 1-D and 2-D constellations at asymptotes followed by the section where we conclude the chapter.

In the fourth and fifth chapter our aim is to deal with the same problem for 1-D and 2-D constellations but at finite SNRs. In the first section of fourth chapter we formulate the optimization problem in terms of SER in fading channel as a convex optimization problem and in the following section we derive the necessary conditions for optimality in the same context. In the third section we analyze the bit error rate optimization problem and in the second last section we show the optimization results to show the lower bounds that can be achieved for 1-D constellations. At the end we conclude the chapter in the last section.

In the first section of fifth chapter we come up with numerical optimization framework for solving SER and BER optimization and in the next section we use optimization procedures to show the optimal constellations and improvements for the case of 8 and 16 point constellations in AWGN channel. In the first section of sixth chapter we analyze the general adaptive transmit and constellation allocation problem followed by convex formulation for the case of 1-D constellations. In the second last section we show the optimization results to show the improvements possible followed by the section on conclusion. In the last chapter we conclude the whole work and describe the important future work which should follow this contribution.

Chapter 2

Literature Survey

In this chapter we first describe the system model used in the work and then go on to the general formulation of the problem. In the next section we explain in detail the past work in the literature on particular cases of the problem and highlight the gaps that are there in the current understanding of optimal constellations.

2.1 System Model

In this work we consider a synchronous digital communication system, where the transmitter encodes the message to be sent in the form of symbols chosen from a signal constellation. Signal constellation $S = \{s_1, s_2, ..s_N\}$, representing the set of N symbols where $s_i \in \Re^m \;\; m \;\; \in \;\; \{1, 2\} \;\; \forall \; i$ is either 1 or 2 dimensional and $\{c(1), c(2), ..c(N)\}$ represents the bit labeling scheme, $c : \{1, ..N\} \rightarrow \{0, 1\}^k$ here $k = \lceil \log_2 N \rceil$ is the length of the label. The transmitter linearly modulates these symbols on the transmitter pulse $p(t)$ and the narrow band passband message signal, $m_b(t)$ which is sent across the channel is given as

$$m_b(t) = \sum_k (s^I_{r(k)} + js^Q_{r(k)})p(t - kT) \tag{2.1}$$

Here T is the duration of the message symbol and $r : \{1, ...\} \rightarrow \{1, ..N\}$ represents the function to select the symbol, $s^I_{r(K)}$, $s^Q_{r(K)}$ are the in phase and quadrature phase components of the symbol to be sent. The transmit symbol is sent across the channel which can be AWGN or fading. Fading model considered in our work is frequency flat and slow fading and we assume ideal coherent detection at the receiver. In the case of AWGN channel, the receiver performs matched filter detection to decide which symbol was sent. Hence the output of the receiver can be modeled as

$$r_k = s_k + n_k$$

Here $s_k = s_k^I + js_k^Q$ is the symbol point from 2-D constellation and we can assume quadrature phase component to be zero in case of 1-D constellation, and n_k is complex additive white gaussian noise, $\mathcal{CN}(0, \sigma^2)$. In case of fading channel the output at the receiver is given as

$$r_k = hs_k + n_k$$

Here h is the flat fading coefficient and since we assume coherent detection

$$\frac{h^*}{|h|} r_k = |h| s_k + \frac{h^*}{|h|} n_k$$

The decision device at the receiver partitions the 1-D or 2-D space into N decision regions corresponding to each symbol and whenever the received symbol is outside the decision region of the symbol that was sent error occurs. The general expression for symbol error rate assuming equiprobable signalling is

$$\begin{aligned} P_{se}^A(s_1, ..s_N) &= \frac{1}{N} \sum_{k=1}^{N} P_{se}^A(s_1, ..s_N | s_k) \\ P_{se}^f(s_1, ..s_N) &= \frac{1}{N} \sum_{k=1}^{N} \int_0^\infty P_{se}^A(\alpha s_1, ..\alpha s_N | s_k) f_{|h|}(\alpha) d\alpha \end{aligned} \tag{2.2}$$

Here P_{se}^A and P_{se}^f correspond to symbol error in AWGN and fading channel respectively and $P_{se}^A(..|s_i)$, $P_{se}^f(..|s_i)$ are the error probabilities given s_i is sent. On the same lines we can come up with bit error rate. Bit error rate on the same lines is given as,

$$P_{be}^{A}(s_1,..s_N,c(1),..c(N)) = \frac{1}{N\lceil \log_2 N\rceil}\sum_{k=1}^{N}\sum_{j=1}^{N} d(c(k),c(j))P(s_j|s_k) \tag{2.3}$$
$$P_{be}^{f}(s_1,..s_N,c(1),..c(N)) = \frac{1}{N\lceil \log_2 N\rceil}\sum_{k=1}^{N}\int_{0}^{\infty}\sum_{j=1}^{N} d(c(k),c(j))P(\alpha s_j|\alpha s_k)f_{|h|}(\alpha)d\alpha$$

Here P_{be}^{A} and P_{be}^{f} are bit error rate functions in AWGN and fading channel respectively and $P(s_j|s_i)$ is the probability that s_j is detected given s_i is sent.

2.2 General Problem Formulation

In this section we state a general formulation of the problem dealt with in this work. Here we would state the optimization problem in such a way that the objective function P_e^f would take both symbol and bit error rate into account for an N point constellation in both fading and AWGN channel.

Each symbol point s_i can be represented by its coordinates in $\Re^2$ for 1-D/2-D constellation. Error rate (symbol/bit) $P_e^f(s_1,..s_N,c(1),..c(N))$ is dependent on the positions of point, the coding scheme used and distribution of channel gain $\{g=|h|^2, f_G(g)\}$. The general problem can be stated as

$$\begin{aligned} &\min P_e^f(s_1,..s_N,c(1),..c(N)) \\ &\text{subject to } \sum_{i=1}^{N}\|s_i\|_2^2 = c \end{aligned} \tag{2.4}$$

The above problem is relevant as it helps define a limit below which one cannot push the error rate for a fixed size and average power and what signal set to use to achieve this limit. It is important to understand these solutions as we would see later that these can be used in adaptive schemes in which power and signal set can be adapted.

2.3 Past Work

Multilevel Phase Modulation based communication systems were initially developed by Doelz et.al in [6] and were shown to be efficient in bandwidth versus SNR trade-off by Cahn [7]. Cahn analyzed the performance of phase modulated systems in gaussian noise under coherent detection and phase comparison detection technique. In this work itself Cahn gave insights on the idea of combining phase and amplitude modulation and in [1] it was shown that as the number of bits per sample grow the combination utilizes the transmit power more efficiently. Following this the work of Hancock et.al [2] analyzed the performance of two types of transmission systems one where amplitude and phase channels are uncorrelated and the second type where the two streams are dependent and the second is shown to be superior. Work of Camopiano et.al [3] showed the 2 dimensional signal constellation idea and coherent demodulation of in phase and quadrature phase components independently, this is fundamental to digital communications till date.

Finding best signal sets in terms of symbol error rate for constraints like peak and average power were the next set of interesting problems which came into picture. Some initial works in this direction were either heuristic or ad hoc. In [8] Lucky et.al characterized the solution to the symbol error rate optimization problem illustrating that at low SNRs phase modulation is useful and at higher SNRs a combination of both. At low SNRs expression of error rate is shown to decrease with increase in the perimeter of convex polygon enclosing the constellation, showing phase modulated constellations as ideal in that range. In [9] C.Thomas et.al generated a set of 29 constellations with size ranging between 4 and 128 and investigated them for optimum designs. Next [10] proposed a really nice heuristic idea to look at the problem in which points were allowed to take only discrete positions in the plane. This was the first work which formulated this as an optimization problem though an approximate one and solved it efficiently, thus proving to be an effective tool in obtaining near optimum solutions. Each of these works had certain assumption which led to only symmetrical solutions.

The above works were followed by really significant work of Foschini et.al [4]. The problem was looked at from the perspective of optimization formulation. Gradient search based procedures were used to come up with the locally optimal solutions. An important assumption of the work was asymptotic (large signal to noise ratio) which meant that problem was unanswered at any finite SNR value. The main problem while solving this problem was the fact that the objective function does not have a closed form solution in terms of any general arrangement of points. In [11] Craig proposed an elegant method for expressing symbol error rate function for any arbitrary shaped decision region (arbitrary polygon) as a sum of one dimensional finite integrals. It is important to understand that for a fixed relative arrangement of points, exact symbol error rate expression can be written since the shape of decision regions are known, implying that analyzing symbol error rate as a function of SNR for a fixed relative arrangement is relatively easier. [12] exploits this and show convexity of SER as a function of SNR under certain conditions. Actual difficulty come when SNR is fixed and relative arrangement can be varied, in such a scenario the objective function cannot be written in closed form. Interestingly the expression for symbol error rate for a 1 dimensional constellation does have a closed form for any arrangement and Makowski et.al in [5] come up with necessary conditions for optimal constellation under any general energy constraint in AWGN channel.

A surge in the area of wireless communications gave rise to the same problem in the context of fading channels and recent works in this direction focus on considering a family of constellations defined by certain parameters and then optimize those parameters at a fixed SNR. [13] came up with new expressions for various signal sets in context of fading, which helped optimize the parameters of these signal sets and come up with optimal ring ratios for signal sets. Recent works like [14] and [15] come up with regular structures which perform better than QAM and have less decoding complexity but of course are not optimum. [16] generalize the QAM to a θ QAM family, arrive at exact symbol error expression and the parameter θ is optimized, basically this general family captures symmetrical constellations like triangular QAM and square QAM in a nice fashion. The main limitation of the work is that it gives optimal solutions within this symmetrical family of constellations.

Interestingly the problem of optimization of constellations (1-D/2-D) with respect to bit error rate has not received as much attention as symbol error rate. The problem in that case is not only of designing the constellation but also the labeling scheme which makes it fairly difficult. But there are works in literature which deal with proving optimal labeling for a fixed relative arrangement of symbols. [17] shows that gray coding is optimal for PAM,QAM and PSK type of constellations. In [16] the authors arrive at close to accurate expressions for bit error rate and show that SQAM is optimal within θ QAM family in low SNR region and $\theta \approx 65^\circ$ in high SNR range for an AWGN channel.

2.4 Conclusion

At the end of this section we can conclude that we still do not know " the best" 2-D signal constellations in terms of SER/BER at a finite SNR in both AWGN and fading channel. Also the best 1-D signal constellations in terms of SER/BER in a fading channel are unknown. In the next chapter we would look at the problem for optimizing both 1-D and 2-D constellations in both AWGN and fading channel at $SNR \to \infty$.

Chapter 3

Optimizing Signal Constellations at Asymptotically high SNRs

In this chapter we deal with optimization problem described in the previous chapter but at asymptotically high SNR values. Necessary conditions for optimality are proposed and conditions proposed by Foschini et.al are shown to be inaccurate.

In the first section we show the formulation of the problem at asymptotes for 1-D and 2-D constellations in both AWGN and fading channel. In the next section we propose necessary conditions for optimality of 2-D constellations in AWGN channel followed by the section where we show the counterexample to Foschini's conditions. In the section to follow we show optimization results for both 1-D and 2-D constellations. We conclude the chapter in the last section.

3.1 Problem Formulation at Asymptotes

We state two necessary conditions in the below given lemmas which every optimal constellation is bound to satisfy irrespective of SNR,

Lemma 3.1. $s^* = \{s_1^*, s_2^*, s_3^*, ...s_N^*\}$ is a solution to the general optimization problem in 2.5 only if the centroid $s_c = \frac{1}{N}\sum_{i=1}^{N} s_i^* = 0$.

Proof. Firstly we state that as the minimum value of the error rate $P_e(s_1, ..s_N, c(1), ..c(N))$

decreases with c, here c is the total power constraint. This can be seen from [12], where SER is shown to decrease with SNR for any arbitrary constellation and on the same lines same can be said for BER. Suppose $\{s_1^*, s_2^*, ...s_N^*\}$ is a minimum and $s_c = \frac{1}{N}\sum_{i=1}^{N} s_i^*$ is the centroid and $T(s_1, ..s_N)$ is the function characterizing total energy in a constellation.

$$\begin{aligned} T(s_1^*, s_2^*, ..s_N^*) &= \sum_{i=1}^{N} \|s_i^*\|_2^2 = \sum_{i=1}^{N} \|s_i^* - s_c + s_c\|_2^2 \\ &= \sum_{i=1}^{N} \|s_i^* - s_c\|_2^2 + N\|s_c\|_2^2 + 2\sum_{i=1}^{N}(s_i^* - s_c)^T s_c \end{aligned} \tag{3.1}$$

In the above the cross term becomes 0 since s_c is the centroid. Assume $s_c \neq 0$, this would mean that the total power in the equation (3.1) can be reduced by translating such that the relative arrangement of the points is same and $s_c' = 0$, here s_c' is the centroid of the translated constellation. Since the relative arrangement of points is same the symbol error rate is same [12]. This means same SER is obtained at a lesser power c value. This combined with the fact that optimal SER decreases with c leads that $\{s_1^*, ..s_N^*\}$ cannot be optimal. Therefore for the solution to be optimal s_c has to be 0. ■

Lemma 3.2. If a constellation $\{s_1^*, s_2^*, ..s_N^*\}$ is optimal solution to 2.5 then the derivative of $P_e(s_1, s_2, ..s_N, c(1), ..c(N))$ is zero on the surface $\sum_{i=1}^{N} \|s_i\|^2 = c$ at $\{s_1^*, s_2^*..s_N^*\}$ and is proportional to $(s_1^*, ...s_N^*)$.

Proof. The proof of the above follows straightaway from KKT conditions.

$$\mathcal{L}(s_1, ..s_N) = P_e(s_1, ..s_N, c(1), ..c(N)) + \lambda(T(s_1, ..s_N) - c)$$

$$\nabla\mathcal{L}(s_1^*, ..s_N^*) = \nabla P_e(s_1^*, ..s_N^*, c(1), ..c(N)) + \lambda\nabla T(s_1^*, ..s_N^*) = 0$$

$$\nabla P_e(s_1^*, ..s_N^*, c(1), ..c(N)) = -2\lambda[s_1^*, ..s_N^*]$$

■

Having described the general optimization problem and the necessary conditions for optimality we now focus on coming up with the optimal geometries at asymptotically high SNR values. We now give a lemma which gives a general relation between symbol and bit error rate, expression given below from 2.2 and 2.4,

$$P_{se}(s_1,..s_N) = \frac{1}{N}\sum_{k=1}^{N} P_{se}(s_1,..s_N|s_k)$$

$$P_{be}(s_1,..s_N,c(1),..c(N)) = \frac{1}{N\lceil\log_2 N\rceil}\sum_{i=1}^{N}\sum_{j=1}^{N} d(c(j),c(i))P_{se}(s_j|s_i)$$

Here P_{se} is the symbol error rate $P_{se}(..|s_j)$, is symbol error rate given s_j is sent, $P_{se}(s_j|s_i)$ is the probability that s_j is detected given s_i is sent.

Lemma 3.3. The bounds on Bit error rate function $P_{be}(s_1,s_2,..s_N,c(1),..c(N))$ in terms of symbol error rate P_{se}
$\frac{1}{\lceil\log_2 N\rceil}P_{se}(s_1,..s_N) < P_{be}(s_1,s_2,..s_N,c(1),..c(N)) < P_{se}(s_1,..s_N)$

Proof. $P_{se}(s_1,s_2,..s_N) = \frac{1}{N}\sum_{i=1}^{N}\sum_{j\neq i}^{N} P_{se}(s_j|s_i)$ and $d(c(i),c(j)) \leq \lceil\log_2 N\rceil$. From this we have

$$P_{be}(s_1,s_2,..s_N,c(1),..c(N)) = \frac{1}{N\lceil\log_2 N\rceil}\sum_{i=1}^{N}\sum_{j=1}^{N} d(c(j),c(i))P_{se}(s_j|s_i)$$
$$\leq \frac{1}{N\lceil\log_2 N\rceil}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N}\lceil\log_2 N\rceil P_{se}(s_j|s_i) \leq P_{se}(s_1,s_2...s_N)$$

So the upper bound in the lemma is established. Next we see that since $d(c(i),c(j)) \geq 1$, since $i \neq j$ which would imply that,

$$\frac{1}{N\lceil\log_2 N\rceil}\sum_{i=1}^{N}\sum_{j=1}^{N} d(c(i),c(j))P_{se}(s_j|s_i) \geq \frac{1}{N\lceil\log_2 N\rceil}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} P_{se}(s_j|s_i)$$
$$= \frac{1}{\lceil\log_2 N\rceil}P_{se}(s_1,..s_N)$$

So this completes the proof. ∎

3.1.1 Problem Formulation For 1-D Constellation

Let $S = \{s_1, s_2, ..s_N\}$ be the 1-D constellation with each point in $\Re$. Let $P_{se}(s_1, s_2, ..s_N)$ be the symbol error rate and $\{s_{[1]}, s_{[2]}....s_{[N]}\}$ is sorted in the increasing order of value. For the case when the channel has no fading and there is only additive white gaussian noise of variance σ we have

$$P_{se}(s_1, s_2, ..s_N) = \frac{2}{N} \sum_{i=1}^{N-1} Q(\frac{s_{[i+1]} - s_{[i]}}{2\sigma}) \tag{3.2}$$

Here Q is the standard Q function, $Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{t^2}{2}} dt$ On the same lines as in 2.2 we get the expression in flat fading channel whose gain $|h|^2 = g$ is distributed as $f_G(g)$

$$P_{se}(s_1, ..s_N) = \frac{2}{N} \sum_{i=1}^{N-1} \int_0^\infty Q(\frac{\sqrt{g}(s_{[i+1]} - s_{[i]})}{2\sigma}) f_G(g) dg$$

P_{se} at asymptotes $\sigma^2 \to 0$, we can approximate $Q(x) \approx \frac{1}{2} e^{-\frac{x^2}{2}}$

$$P_{se}(s_1, ..s_N) \approx \frac{1}{N} \sum_{i=1}^{N-1} \int_0^\infty e^{-\frac{g(s_{[i+1]} - s_{[i]})^2}{8\sigma^2}} f_G(g) dg = \frac{1}{N} \sum_{i=1}^{N-1} M_f(-\frac{(s_{[i+1]} - s_{[i]})^2}{8\sigma^2})$$

Here M_f is the moment generating function of f and the symbol error rate at asymptotes is sum of MGF of distance between adjacent points squared. The above approximation can be further tightened using the expression from the work of Chiani et.al. [18]

$$Q(x) = \frac{1}{12} e^{-\frac{x^2}{2}} + \frac{1}{4} e^{-\frac{x^2}{3}}$$

to get

$$P_{se}(s_1, ..s_N) \approx \frac{1}{N} \sum_{i=1}^{N-1} (\frac{1}{6} M_f(-\frac{(s_{[i+1]} - s_{[i]})^2}{8\sigma^2}) + \frac{1}{2} M_f(-\frac{(s_{[i+1]} - s_{[i]})^2}{12\sigma^2})) \tag{3.3}$$

3.1.2 Problem Formulation For 2-D Constellation

Let $S = \{s_1, s_2, ..s_N\}$ be the 2-D constellation with each point in $\Re^2$. Let $P_{se}(s_1, s_2, ..s_N)$ be the symbol error rate and $\{D_1, ..D_N\}$ be the N decision regions obtained from the Voronoi diagram of the N points. For the case of AWGN channel the expression goes as follows,

$$P_{se}(s_1, ..s_N|s_i) = \int_{(x,y)\in D_i^c} \frac{1}{2\pi} e^{-\|(x,y)-s_i\|_2^2)} dxdy$$
$$P_{se}(s_1, s_2, ..s_N) = \frac{1}{N}\sum_{i=1}^{N} \int_{(x,y)\in D_i^c} \frac{1}{2\pi} e^{-\|(x,y)-s_i\|_2^2)} dxdy$$

Again on the same lines as 2.2 we get the expression in flat fading channel as

$$P_{se}(s_1, s_2, ..s_N) = \frac{1}{N}\sum_{i=1}^{N} \int_0^\infty \int_{(x,y)\in D_i^c} \frac{1}{2\pi} e^{-\|(x,y)-\sqrt{g}s_i\|_2^2)} f_G(g) dxdydg \tag{3.4}$$

At asymptotes $\sigma^2 \to 0$, we can say that $P_{se}(s_1, ..s_N|s_k)$ and D_k are determined by $\min_{j\neq k} \|s_j - s_k\|$ and can be approximated as

$$P_{se}(s_1, s_2, ..s_N|s_k) \approx \int_0^\infty \exp\Bigg(-g\frac{\min_{j\neq k}\|s_k - s_j\|^2}{8\sigma^2}\Bigg) f_G(g)dg = M_f(-\frac{\min_{j\neq k}\|s_k - s_j\|^2}{8\sigma^2})$$

Here M_f is moment generating function corresponding to distribution f. So the objective function is simply $P_{se}(s_1, ..s_N) = \sum_{k=1}^{N} M_f(-\frac{\min_{j\neq k}\|s_k - s_j\|^2}{8\sigma^2})$ For the case of AWGN channel the distribution f is $\delta(g-1)$ and the $M_f(t) = e^t$, So the error expression is is

$$P_{se}(s_1, s_2, ..s_N) \simeq \frac{1}{N}\sum_{k=1}^{N} \exp\Bigg(-\min_{j\neq k}\frac{\|s_k - s_j\|^2}{8\sigma^2}\Bigg)$$
$$P_{se}(s_1, s_2, ..s_N) \simeq$$
$$\frac{1}{N}\exp\Bigg(-\min_{j\neq k^*}\frac{\|s_{k^*} - s_j\|^2}{8\sigma^2}\Bigg)(1 + \sum_{k\neq k^*}\exp\Bigg(\min_{j\neq k^*}\frac{\|s_{k^*} - s_j\|^2}{8\sigma^2} - \min_{j\neq k}\frac{\|s_k - s_j\|^2}{8\sigma^2}\Bigg))$$

In the above, k^* is the index of that symbol whose nearest neighbor's distance is lesser than or equal compared to any other symbol. So at asymptotes we can say that P_{se} would scale exponentially as the term outside the bracket as the terms inside would go to zero. This leads to

$$P_{se}(s_1, s_2, ..s_N) \sim \exp\left(- \min_{j \neq k^*} \frac{\|s_j - s_{k^*}\|^2}{8\sigma^2} \right) \tag{3.5}$$

From Lemma 3.3 we can say that the bit error rate at asymptotes is

$$\frac{1}{\lceil \log_2 N \rceil} \exp\left(- \min_{j \neq k^*} \frac{|s_j - s_{k^*}|^2}{8\sigma^2} \right) < P_{be}(s_1, ..s_N, c(1), ..c(N)) < \exp\left(- \min_{j \neq k^*} \frac{|s_j - s_{k^*}|^2}{8\sigma^2} \right)$$

Since the bit error rate function is sandwiched from both sides by symbol error rate, so the rate at which it goes to 0 is decided by pairwise minimum distance among all possible pairs and would not depend on the labeling scheme c. So the equivalent optimization problem is ,

$$\begin{aligned} &\text{Max} \min_{j \neq k} \|s_j - s_k\| \\ &\text{subject to} \sum_{i=1}^{N} \|s_i\|^2 = c \end{aligned} \tag{3.6}$$

Now we look at the case of bit error rate in fading channels. From Lemma 3.3 we can write

$$\frac{1}{\lceil \log_2 N \rceil} \sum_{k=1}^{N} M_f(- \min_{i \neq k} \frac{\|s_i - s_k\|^2}{8\sigma^2}) < P_{be}(s_1, ..s_N, c(1), ..c(N)) < \sum_{k=1}^{N} M_f(- \min_{i \neq k} \frac{\|s_i - s_k\|^2}{8\sigma^2}) \tag{3.7}$$

Result 3.1. $\lim_{a \to \infty} M_f(-a) = 0$

$M_f(a) = \int_0^\infty e^{-ax} f(x)dx = \int_0^\epsilon e^{-ax} f(x)dx + \int_\epsilon^\infty e^{-ax} f(x)dx$. For any $\epsilon > 0$ the second term $\int_\epsilon^\infty e^{-ax} f(x)dx < e^{-a\epsilon} \int_\epsilon^\infty f(x)dx < e^{-a\epsilon}$ and the first term $\int_0^\epsilon e^{-ax} f(x)dx < \int_0^\epsilon f(x)dx < \sup_{x \in [0,\epsilon]}(f(x))\epsilon$. Assuming $f(x)$ to be bounded we can say that both the terms approach 0. This completes the result.

Using the above result and (3.7) we get that optimizing bit error rate at asymptotic SNR is equivalent to

$$\min \sum_{k=1}^{N} M_f\left(-\frac{\min_{i\neq k} \|s_i - s_k\|^2}{8\sigma^2}\right) \tag{3.8}$$
$$\text{subject to} \sum_{k=1}^{N} \|s_k\|^2 = c$$

In the next section we analyze the problem in 3.6 and come up with necessary conditions for optimality.

3.2 Necessary Conditions for Optimality of 2-D Constellation in AWGN Channel

Here we come up with necessary conditions which imply symmetry in the optimal constellation. We start with stating the following result which would be used later.

Result 3.2. Maximum of the minimum distance from the neighbors which is a solution to (3.6), is a strictly increasing function of c.

The proof to the above result is obvious, as c is increased one can proportionally increase the power in each symbol leading to increased separation between each of them.

Now we come up with necessary conditions for optimality.

Theorem 3.1. If $S^* = \{s_1^*, s_2^*, ...s_N^*\}$ is a solution to the above problem in (3.6), then each symbol s_i^* would have a nearest neighbor which would attain the optimal maximum distance.

Proof. Let $S^* = \{s_1^*, s_2^*, ...s_N^*\}$ be a solution to (3.6) and $\max \min_{k\neq j} \|s_k - s_j\| = \alpha(c)$ is the optimal maximal distance which increases with c.

Suppose we assume that there exists an s_i^* for which $\min_{j\neq i} \|s_j^* - s_i^*\| > \alpha(c)$. As a result we can reduce magnitude of s_i^* by a factor of $m < 1$ to the point satisfying

$\min_{j \neq i} \|s_j^* - ms_i\| \geq \alpha(c)$. This would reduce $\|s_i\|$ and this in the magnitude of s_i would lead to a decrement in total power $c - \delta c$, this would mean that $\alpha(c - \delta c) = \alpha(c)$. This leads to a contradiction as we know that α is a strictly increasing function. ■

Theorem 3.2. If $S^* = \{s_1^*, s_2^*, ..s_N^*\}$ is a solution to (3.6), then each point s_i^* will have at least two nearest neighbors which attain the same optimal maximum distance and the two neighbors lie on the opposite side of line joining origin to s_i^*

Proof. Let $s_i^* = (r_i^*, \theta_i^*)$ and s_j^* be the neighbor located at the least distance $\alpha(c)$ from it. Also let us assume that $s_j^* = (r_j^*, \theta_j^*)$ be the only neighbor which is nearest to it. The set of remaining points consist of points apart from j is $S_{i-j} = \{s_i^*\}^c \cap \{s_j^*\}^c$ and $\min_{s_k \in S_{i-j}} \|s_i^* - s_k^*\| = \beta(c)$. We can assume $\theta_i^* - \theta_j^* > 0$ without loss of generality. We can write $\alpha(c)$ as $\alpha(c, \theta_i)$, a function θ_i

$$\alpha(c, \theta_i)^2 = r_i^{*2} + r_j^{*2} - 2r_i^* r_j^* \cos(\theta_i - \theta_j^*)$$
$$\frac{d\alpha(c, \theta_i^*)^2}{d\theta_i} = 2r_i^* r_j^* \sin(\theta_i^* - \theta_j^*) > 0$$

Since $\beta(c, \theta_i^*) > \alpha(c, \theta_i^*)$ and change $\delta\theta_i^* > 0$ is such the minimum distance $\|s_i - s_j^*\|$ increases and $\beta(c, \theta_i)$ should not decrease below $\alpha(c, \theta_i^*)$. Upon this rotation s_i would have no nearest neighbor at the optimal distance implying that $\|s_i\|$ can be reduced like in theorem 3.1. This would imply that the optimal minimum distance can be increased. This is a contradiction and so the assumption that the point will have only one minimum distance neighbor cannot be true. So, the point needs to have at least two nearest neighbors and if both are on the same side of the line from origin to s_i^* then again a rotation would lead to contradiction. ■

These necessary conditions go on to show that optimal constellations are symmetric at asymptotically high SNR values. This fact is not coherent with the work of foschini and the next section deals with this claim.

3.3 Counterexample to Foschini's Conditions

3.3.1 Error in Foschini's Work

Equation 21 in [4] gives the expression for the gradient $g_k = \sum_{i \neq j} e^{-\frac{\|s_i - s_j\|^2}{8N_o}} (\frac{1}{\|s_i - s_j\|^2} + \frac{1}{4N_o}) 1_{s_k - s_i}$

It is given that as $N_o \to 0$ then $g_k = \alpha \sum_{i \in N(k)} 1_{s_i - s_k} = s_k$ and let $\{s_1, ..s_N\}$ be a local minimum of the expression in (8) in [4]. The above necessary condition is not correct because the nearest neighbors of s_k are not at the same distance from it and as $N_o \to 0$ the term in the exponential of the derivative is not the same for every nearest neighbor which is assumed in the necessary condition approximation. The correct way to look at the problem is at asymptotes when $N_o \to 0$, it is important to see that any arrangement would give $SER \to 0$ but the rate at which these constellations is the deciding factor. The rate is decided by $\min_{i \neq j} \|s_i - s_j\|$ and the constellation with highest rate is said to be asymptotically optimal. Basically at the asymptotes the objective function is $\min_{i \neq j} \|s_i - s_j\|$ and this being non differentiable makes the approach of taking gradient infeasible, so we take a different approach described below.

Now we would come up with the asymptotically optimal 5 point constellation and then show that it does not satisfy the necessary conditions given by foschini et.al. Any possible arrangement of 5 points falls into one of the following category 1. All collinear 2. Triangle with 2 points inside 3. Convex quadrilateral with one point inside 4. Convex pentagon

First we would see that using theorem 3.1 and 3.2 we would come up with symmetrical families which satisfy the necessary conditions and from these we arrive at the optimum solution.

Theorem 3.3. For a 5 point constellation, no irregular convex pentagonal constellation qualifies as a candidate for optimal constellation and for a convex quadrilateral with one point inside only the family of two equilateral triangles with a common vertex satisfy the necessary conditions in previous section.

Proof. **Convex Pentagon**

Let us consider any 5 point convex polygonal constellation and the points $\{s_1, s_2, ..s_5\}$ in the cyclic order form the convex polygon which means s_1 has edge to s_2 and s_5.

From theorem 3.1 and 3.2 we know each point would attain at least same minimum among the nearest points.

This leads us to say that each point s_i can have both optimal distances 1)attained by adjacent neighbors, 2) one by adjacent one by next to adjacent, 3)both by next to adjacent.

Let us do an exhaustive case by case analysis.

1. Firstly if every point has the nearest neighbors as the adjacent points then we obviously have a regular convex polygon.

2. If at least one point has nearest neighbors as next to adjacent points then we see what happens. Let s_1 has one nearest neighbor in the form of s_3 and other as s_4. So $d_{13} = d_{14} = d_{min}$ as shown in figure 3.5 and lets go on to s_4, one of its nearest neighbor is s_1 and other nearest neighbor can't be s_2 because $d_{34} \geq d_{min}$. Lets assume $d_{42} = d_{min}$ then $\angle s_4s_1s_3 > \frac{\pi}{3}$ using cosine rule in triangle $s_4s_1s_3$. $\angle s_4s_1s_2 < \frac{\pi}{3}$ using cosine rule in triangle in $s_4s_1s_2$. This is not possible as it leads to contradiction.

   (a) If $d_{34} = d_{min}$ then it is clear that $s_1s_3s_4$ form an equilateral triangle. From theorem 1.2, Points s_2 and s_5 their will have nearest neighbors as s_1, s_3 and s_1, s_4 respectively . So we have all the points located on a regular convex polygon.

   (b) If $d_{34} > d_{min}$, the other nearest neighbor of s_4 has to be s_5 as it cannot be s_2 which has to be on the other side of s_1s_3 . The other nearest neighbor of s_5 apart from s_4 has to be s_1. So, $s_1s_4s_5$, $s_1s_2s_3$ form equilateral triangles making the whole all the adjacent distances in the constellation to be same except for d_{34}. If $d_{34} > d_{min}$ the constellation would not be

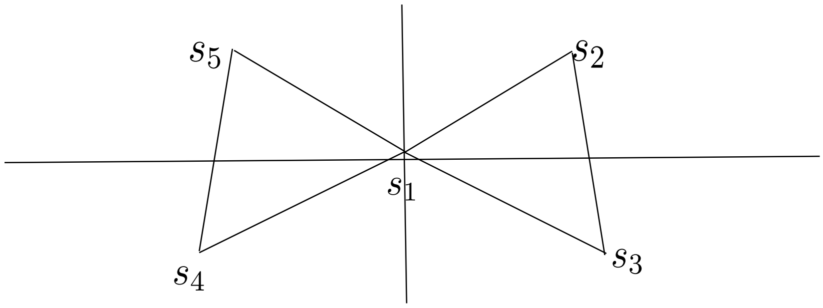

Figure 3.1: Two Equilateral Triangles with one vertex common

convex (prove using two angles from equilateral triangle $\frac{\pi}{3}$ and other > $\frac{\pi}{3}$ total angle > π, therefore $d_{34} = d_{min}$.

3. Now we have to look at the case when s_1 has one nearest neighbor as s_4 and the other as s_2. In this case, if s_2 has s_4 as the nearest neighbor then it would be the same as the case of at least one point having two non adjacent nearest neighbors. Therefore $d_{24} > d_{min}$ and also s_3 cannot have s_5 as nearest neighbor. If s_3 has s_2 and s_4 as nearest neighbors and s_5 can only have s_1 and s_4 as nearest neighbors giving a regular pentagonal structure. If s_3 has s_2 and s_1 as nearest neighbors this combined with s_4 having s_5 and s_1 as nearest neighbors give us regular pentagonal structure.

Convex Quadrilateral with one point inside

Now we move to the case of convex quadrilateral with one point inside. Let $\{s_1, ..s_4\}$ form the convex quadrilateral and s_5 is a point inside. So at least one point has s_5 as nearest neighbor let that point be s_1 and s_1's other nearest neighbor can be adjacent or next to adjacent point. If we choose the next to adjacent point which means s_3 then s_2 will have s_1 as the nearest neighbor which means this case is already considered if we assume only the adjacent point case. So let s_1 have s_2 as the other nearest neighbor. Now s_2 can have s_5 as nearest neighbor forming an equilateral triangle which means the other two points s_4 and s_3, if one of them s_5 as nearest neighbor then we will have another equilateral triangle in the form of $s_4s_3s_5$. Two equilateral triangles with s_5 common and centroid at origin as in figure

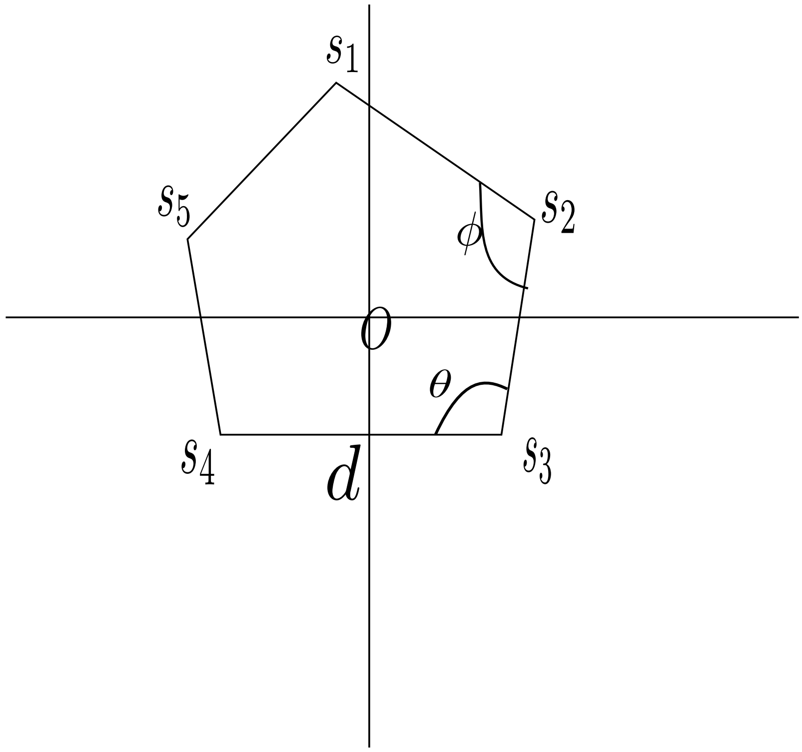

Figure 3.2: Regular Pentagon, parameters θ and ϕ

3.1, is the family of constellations which satisfy necessary conditions. It can happen s_3 and s_4 have nearest neighbors as the adjacent points s_1 and s_2 as neighbors and $d_{34} = d_{min}$ such a case is not possible as (contradiction sum of angles). If s_2 had s_3 as the other nearest neighbor with $d_{25} > d_{min}$ and s_3 has option of s_4 and s_5 in both these cases a rhombus would be formed which would not satisfy the condition that s_5's distance to the vertices is more than edge length.

Based on similar lines, we can show that the cases for triangle with two point and collinear one do not qualify as optimal candidates. ∎

Basically in this lemma we have applied necessary conditions derived in theorem 3.1 and 3.2 to come up with the a family of convex pentagons and two equilateral triangles with a common vertex and centroid at origin which are possibly solution of (3.6). Now we have to find the optimal member among the family of regular pentagons.

In figure 3.2 we see a regular pentagon with side d and the family is parameterized by θ and ϕ with centroid at O. First let us try and find the minimum for the case

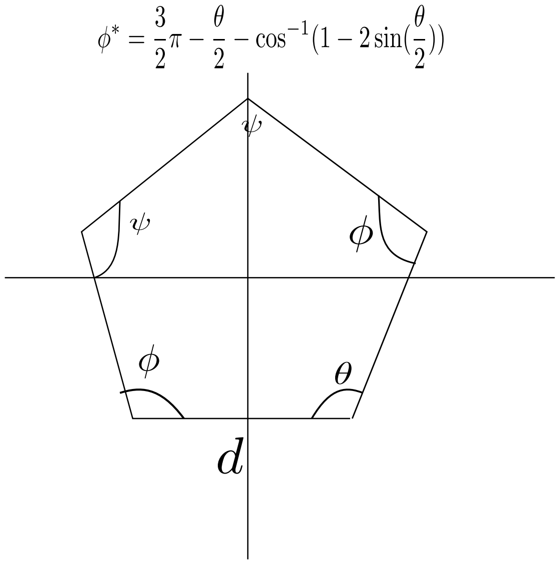

Figure 3.3: Regular Pentagonal, $\phi = \phi^*$

when θ is fixed and ϕ is varied keeping the total power fixed and the centroid at the origin. In general d which is the minimum distance is a function of θ, ϕ and c. But when only ϕ is varied , it is $d(\phi)$. When $\phi = \phi^*$ given below, the constellation is given in figure 3.3

$$\phi^* = \frac{3}{2}\pi - \frac{\theta}{2} - \cos^{-1}(1 - 2\sin(\frac{\theta}{2})) \tag{3.9}$$

From the symmetry of the situation we can see that $d(\phi^* + \gamma) = d(\phi^* - \gamma)$ which means the derivative is zero at ϕ^* for a fixed θ and the function d is decreasing on both sides of ϕ^* and this claim can be shown by plotting d as a function of ϕ for a fixed θ.

So now the problem reduces to comparing these minima for different values of ϕ^*.

For simplicity of analysis we would use the rotated version shown in figure 3.4 parameterized by α

So what we need is the expression for $d(\alpha)$ $0 \leq \alpha \leq \frac{\pi}{3}$.

$$d(\alpha) = \sqrt{\frac{c}{\frac{1}{2} + 2\cos^2(\alpha) + \frac{4}{5}\sin^2\alpha + \frac{6}{5}(1 - (\cos(\alpha) - \frac{1}{2})^2) + \frac{4}{5}\sin(\alpha)\sqrt{(1 - (\cos(\alpha) - \frac{1}{2})^2)}}}$$

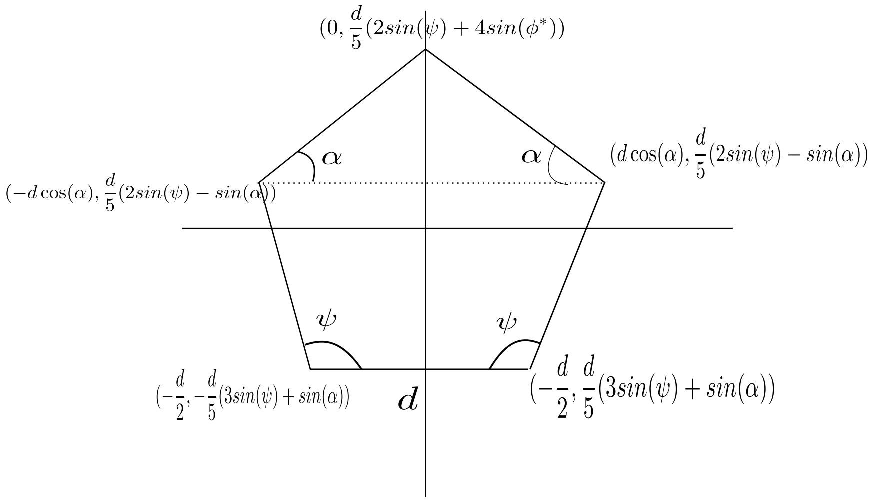

Figure 3.4: Regular Pentagonal with parameter α

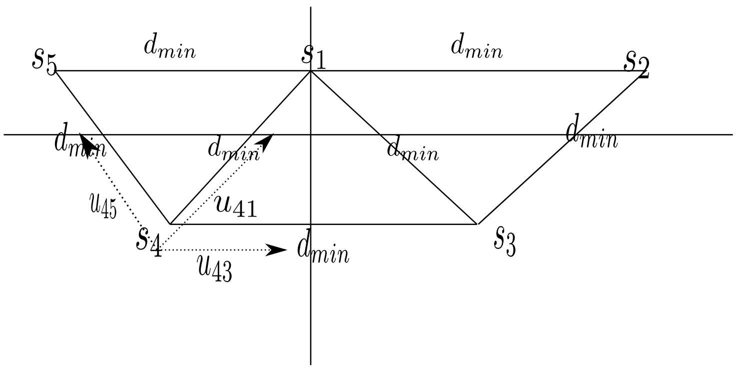

Figure 3.5: Optimal Constellation

From the plot and derivative of this function one can say that it takes a minimum value at $\alpha = 0$. So from this we have the optimum constellation as shown above in figure 3.5.

Among the family of equilateral triangles sharing a common vertex we can say that the optimal constellation shown above will have least energy for a fixed distance. If s_1 is origin then each of these have same energy but we need to have centroid at the origin, so the arrangement needs to be translated to have centroid as the origin. So the arrangement which needs to be translated the maximum distance would have least energy, implying the optimal constellation shown above will have least energy.

The above optimal constellation is a counterexample to Foschini's conditions because if we add the unit vectors in the direction of nearest neighbors of s_4, we do not get the resultant in the direction of s_4. $u_{34} + u_{45} + u_{41}$ points in u_{41}.

3.4 Optimization Results

For the case of 1-D constellation,

Result 3.3. Uniform N point PAM is globally optimal at asymptotic SNR values in terms of SER and BER in AWGN channel, best possible labeling scheme being binary reflected gray code.

Proof. First we would show that no constellation in which the distance between the adjacent points is not equal is not a possible solution to the above problem.

Let us assume that $\bar{s} = \{s_1, s_2, ..s_N\}$ be a solution to the above maximization problem in which at least one point is not located at the same distance from its adjacent points. Let s_i be that point and let $d_{i,i+1} < d_{i,i-1}$. If this is the case then this would mean that magnitude of s_i can be reduced at least till the point where the two neighbors are at the same distance from i. This would mean that the same maximum distance can be achieved at a lower power implying that the $\bar{s}$ is not a minimum from Result 3.2.

So this establishes the fact that only an arrangement of points in which inter symbol distance between nearest adjacent neighbors is same can be optimal. This condition of equal distance combined with the fact that centroid of the set of points is at origin (in Lemma 3.1) gives uniform PAM as the unique optimal solution.

Since we know that uniform N-PAM is the optimal constellation but we have not commented on labeling for BER. In order to have a labeling which helps BER achieve the lower bound given in lemma 3.3, binary reflected gray coding scheme is the best amongst all possible schemes [17]. ■

1-D constellation in Rayleigh Fading channels, the objective at asymptotes is proportional to $\sum_{i=1}^{N-1} \frac{1}{(s_{[i+1]}-s_{[i]})^2}$. Interestingly optimizing this objective under the power constraint gives us non uniform constellations as the optimal solutions. For 8 point constellation the optimal solution in terms of SER when total power is fixed to unity is $\{-0.556, -0.371, -0.216, -0.071, 0.071, 0.216, 0.371, 0.556\}$. For the case of

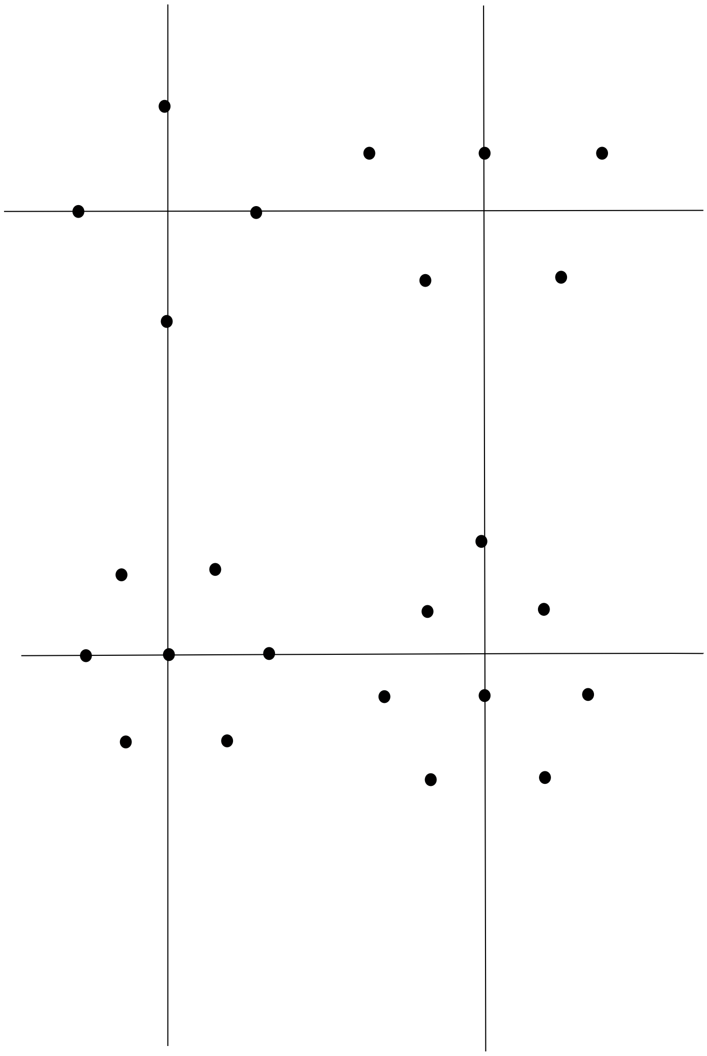

Figure 3.6: Lattice of Equilateral Triangles, Optimal constellations at asymptotes in both AWGN and Rayleigh fading channels

BER, as seen in the previous sections, the same constellation with binary reflected gray coding would be optimal.

For the 2-D constellation in AWGN and Rayleigh fading channel, the optimal constellations in terms of SER/BER form a lattice of equilateral triangles as shown in the figure 3.6.

3.5 Conclusion

In this chapter we have analyzed the BER/SER optimization at asymptotes for 1-D/2-D constellations in both AWGN and fading channel. At asymptotes it is shown that the optimization in terms of bit error rate is equivalent to symbol error rate for both AWGN and fading channels. 2-D optimal constellations in both AWGN and rayleigh fading channel are shown to form a lattice of equilateral triangles. Necessary conditions arrived at by Foschini are shown to be inaccurate and alternate necessary conditions are given. In the following chapters our aim is to solve the same problem at finite SNR values.

Chapter 4

Optimizing 1-D Constellations in terms of Error Rate at Finite SNR

In this chapter our aim is to come up with best 1 dimensional constellations in terms of both SER/BER in both AWGN and fading channels. We also come up with necessary conditions for optimality in terms of symbol error rate in both AWGN and fading channels.

In the first section we deal with convex formulation of optimization problem in terms of symbol error rate in both AWGN and fading channel. Next we go on to come up with necessary conditions for optimality for the same. In the following section we analyze the bit error rate optimization problem . Moving on in the second last section we come up with optimal constellations and show the improvements in comparison to uniform PAM and we follow this section with conclusion to the chapter.

4.1 Optimization in terms of Symbol Error Rate

The closed form symbol error rate expression for any possible arrangement can be written unlike the 2 dimensional case. Let $P_{se}(s_1, s_2, ..s_N)$ be the symbol error rate and $\{s_{[1]}, s_{[2]}....s_{[N]}\}$ is sorted in the increasing order of value. For the case when the channel has no fading and there is only additive white gaussian noise of variance σ

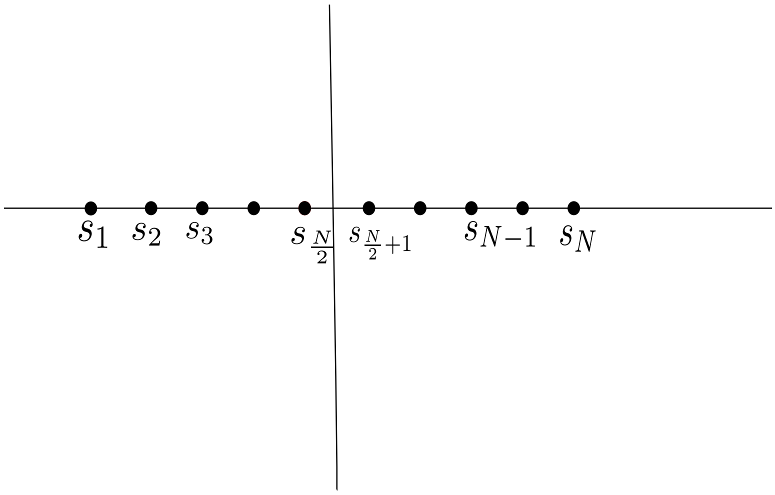

Figure 4.1: Uniform N PAM

we have from equation 3.2

$$P_{se}(s_1, s_2, ..s_N) = \frac{2}{N}\sum_{i=1}^{N-1} Q(\frac{s_{[i+1]} - s_{[i]}}{2\sigma}) \tag{4.1}$$

Here Q is the standard Q function. Our aim is to minimize the above given a fixed power as in (3.6).

The optimal solution to the above problem has been characterized in the literature by Makowski et.al [5]. Makowski et.al establish that optimal solution need to be symmetric about the origin and the inter symbol distance increase as we move away from the origin. Our aim here is to actually derive the optimal solution and characterize based on signal to noise ratio. We then deal with the problem in a general fading channel case. Having done a complete analysis of optimal solution for symbol error rate in fading channels we would go the analysis of bit error rate.

Let us consider when the channel is fading with additive noise component. Signal Received $y = hs_i + n$ where n is additive white gaussian noise with variance σ^2 and h is the complex fading coefficient. $|h|$ is a random variable whose distribution is $f_{|h|}(\alpha)$ and the average symbol error probability is given as follows from equation 3.3,

$$P_e^f(s_1, s_2, ...s_N) = \frac{2}{N}\sum_{i=1}^{N}\int_0^\infty Q(\frac{\alpha(s_{i+1} - s_i)}{2\sigma}) f_{|h|}(\alpha) d\alpha$$

We can reformulate the optimization problem as a convex optimization problem, but before that let us show that $g(d) = \int_0^\infty Q(\alpha d) f_{|h|}(\alpha) d\alpha \; ; d > 0$ is a convex function on $d > 0$

$$g''(d) = \int_0^\infty Q''(\alpha d) f_{|h|}(\alpha) d\alpha = \int_0^\infty \frac{\alpha^3 d}{\sqrt{2\pi}} e^{-\frac{(\alpha d)^2}{2}} f_{|h|}(\alpha) d\alpha$$

Since $g''(d) > 0 \; ; d > 0$ which would mean that the g is convex on $d > 0$. This helps us formulate the minimization problem as follows

$$P_e^{'f}(d_1, d_2, ..d_N) = \frac{2}{N} \sum_{i=1}^{N} \int_0^\infty Q(\frac{\alpha d_i}{2\sigma}) f_{|h|}(\alpha) d\alpha I(d_i) = \frac{2}{N} \sum_{i=1}^{N} g(d_i) I(d_i)$$

$$\min P_e^{'f}(d_1, d_2, ..d_N) \tag{4.2}$$

$$\text{subject to} \sum_{i=1}^{N} (s_1 + \sum_{k=1}^{i-1} d_k)^2 \leq c \tag{4.3}$$

Now we have the above problem formulated as a convex problem which implies that any local minimum should be a global minimum as well. Based on similar lines as previous section on AWGN we come up with some necessary conditions for optimality.

4.1.1 Necessary Conditions For Optimality

Lemma 4.1. For a constellation to be optimal, it has to be symmetric which means for every symbol $s_i > 0$ there is a point $s_j < 0$ such that $|s_j| = |s_i|$

Proof. Let us assume that $\bar{s} = \{s_1, s_2, ...s_N\}$ is optimal and such that $\{s_1 < s_2, ... < s_N\}$ and $\{s_j - s_i = d_{j,i} \; ; j > i\}$. From the expression of symbol error rate we can also justify that $\bar{s}' = \{-s_N, -s_{N-1} .... - s_1\}$ should give the same error which means that this also is optimal.

Let us construct another constellation $\bar{s}^* = \frac{\bar{s}+\bar{s}'}{2}$ and $|\bar{s}^*| \leq \frac{|\bar{s}|+|\bar{s}^*|}{2} = |\bar{s}|$. The power in $\bar{s}^*$ is less than $\bar{s}$.

$$P_e^f(s_1,..s_N) = \frac{2}{N}\sum_{i=1}^{N} g(d_{i+1,i}) \tag{4.4}$$

Whereas from Jensen's inequality

$$P_e^f(s_1^*,...s_N^*) = \frac{2}{N}\sum_{i=1}^{N} g(\frac{d_{i+1,i}+d_{N+1-i,N-i}}{2})$$
$$\leq \frac{2}{N}\sum_{i=1}^{N}\frac{1}{2}g(d_{i+1,i})+\frac{1}{2}g(d_{N+1-i,N-i}) = P_e^f(s_1,s_2,...s_N)$$

Equality being satisfied only when $d_{i+1,i} = d_{N+1-i,N-i}$ which is the case if $\bar{s}$ is symmetric otherwise the constellation is not optimal. ■

Lemma 4.2. For a constellation to be optimal, as we move away from origin the inter symbol distance increases as we move away from origin, which means $d_{i,i-1} < d_{i,i+1}$ when $s_{i-1} > 0$ and $d_{i,i+1} < d_{i,i-1}$ for $s_{i+1} < 0$

Proof. Assume that we have an optimal arrangement $\{s_1, s_2, ..s_N\}$ in which $s_{i-1} > 0$ and $d_{i,i-1} > d_{i,i+1}$. So suppose we change s_i to $s_i' = s_i + \delta s_i$ and to keep the total power constant we change s_N to $s_N' = s_N + \delta s_N$ such that $s_i\delta s_i + s_N\delta s_N = 0$.

$$\frac{\delta P_e(s_1,s_2,..s_N)}{\delta s_i} = (g'(d_{i,i-1}) - g'(d_{i,i+1})) - \frac{s_i}{s_N}g'(d_{N,N-1})$$
$$= \int_0^\infty \left(Q'(\alpha d_{i,i-1}) - Q'(\alpha d_{i,i+1}) - \frac{s_i}{s_N}Q'(\alpha d_{N,N-1})\right) f_{|h|}(\alpha)d\alpha > 0$$

The above holds because $d_{i,i+1} < d_{i,i-1}$ and

$$\left(Q'(\alpha d_{i,i-1}) - Q'(\alpha d_{i,i+1}) - \frac{s_i}{s_N}Q'(\alpha d_{N,N-1})\right) =$$
$$\frac{1}{2}(e^{-\frac{\alpha^2 d_{i,i+1}^2}{2}} + \frac{s_i}{s_N}e^{-\frac{\alpha^2 d_{N,N-1}^2}{2}} - e^{-\frac{\alpha^2 d_{i,i-1}^2}{2}}) > 0$$

So we see that the derivative is positive with respect to change keeping the constraint satisfied which implies that the constellation can't be optimal. ■

Theorem 4.1. For a constellation to be optimal, the increase in the inter symbol distance as we move away from origin cannot be more than a certain limit. If $p(\gamma) = f_{|h|}(\sqrt{\gamma})$ and $P(s) = L(p(\gamma))$ $s > 0$ where L is the Laplace transform if $s_i > 0$ then $d_{i,i-1} < d_{i,i+1} < \sqrt{P^{-1}\big(\frac{i+1}{2i+1}P(\frac{d_{i,i-1}^2}{4\sigma^2})\big)}$ and if $s_i < 0$ then $d_{i,i+1} < d_{i,i-1} < \sqrt{P^{-1}\big(\frac{i+1}{2i+1}P(\frac{d_{i,i+1}^2}{4\sigma^2})\big)}$

Proof. Here also the idea is somewhat similar, first let us take $\{s_1, s_2, ..s_N\}$ to be optimal. Let $s_i > 0$, and here we change s_i and s_{i+1} only keeping the total power fixed, which would mean $s_i\delta s_i + s_{i+1}\delta s_{i+1} = 0$

$$\frac{\delta P_e(s_1, s_2, ..s_N)}{\delta s_i} = g'(\frac{d_{i,i-1}}{2\sigma}) - (1 + \frac{s_i}{s_{i+1}})g'(\frac{d_{i,i+1}}{2\sigma}) + \frac{s_i}{s_{i+1}}g'(\frac{d_{i+1,i+2}}{2\sigma})$$

Since $s_i < s_{i+1}$ and if $g'(\frac{d_{i,i-1}}{2\sigma}) < (1 + \frac{s_i}{s_{i+1}})g'(\frac{d_{i,i+1}}{2\sigma}) \leq \frac{2i+1}{i+1}g'(\frac{d_{i,i+1}}{2\sigma})$ would mean the derivative above can never be zero implying that such an arrangement cannot be optimal.

In the above we used $1 + \frac{s_i}{s_{i+1}} < \frac{2i+1}{i+1}$ and it can be justified as follows. Since we know $d_{i,i+1}$ is larger than preceding inter symbol distances for all $s_j \geq 0$ and $j < i$. This combined with $\sum_{k=j}^{i} d_{k,k+1} = s_i$, here j is the index of first non negative symbol, gives $d_{i,i+1} > \frac{s_i}{i}$. From this we can say that $1 + \frac{s_i}{s_{i+1}} = 1 + \frac{s_i}{s_i + d_{i,i+1}} < \frac{2i+1}{i+1}$

$$\begin{aligned} g'(\frac{d_{i,i-1}}{2\sigma}) &= -\int_0^\infty e^{-\frac{d_{i,i-1}^2\alpha^2}{4\sigma^2}} \frac{\alpha}{2\sigma} f_{|h|}(\alpha)d\alpha \\ &= -\int_0^\infty e^{-\frac{d_{i,i-1}^2\gamma}{4\sigma^2}} \frac{\sqrt{\gamma}}{2\sigma} f_{|h|}(\sqrt{\gamma})\frac{1}{2\sqrt{\gamma}}d\gamma \\ &= \frac{-1}{2\sqrt{2}\sigma}\int_0^\infty e^{-\frac{d_{i,i-1}^2\gamma}{4\sigma^2}} f_{|h|}(\sqrt{\gamma})d\gamma \\ &= \frac{-1}{2\sqrt{2}\sigma}\int_0^\infty e^{-\frac{d_{i,i-1}^2\gamma}{4\sigma^2}} p(\gamma)d\gamma = -\frac{-1}{2\sqrt{2}\sigma}P(\frac{d_{i,i-1}^2}{4\sigma^2}) \end{aligned}$$

Here $P(s) = \int_0^\infty p(\gamma)e^{-s\gamma}d\gamma$ and $P'(s) = -\int_0^\infty \gamma p(\gamma)e^{-s\gamma}d\gamma < 0$ because $p(\gamma) > 0$ implying P is strictly decreasing in $s > 0$.

$$g'(\frac{d_{i,i-1}}{2\sigma}) < \frac{2i+1}{i+1} g'(\frac{d_{i,i+1}}{2\sigma})$$
$$\frac{-1}{2\sqrt{2}\sigma} P(\frac{d_{i,i-1}^2}{4\sigma^2}) < \frac{-1}{2\sqrt{2}\sigma} \frac{2i+1}{i+1} P(\frac{d_{i,i+1}^2}{4\sigma^2})$$
$$\frac{d_{i,i+1}^2}{4\sigma^2} > P^{-1}(\frac{i+1}{2i+1} P(\frac{d_{i,i-1}^2}{4\sigma^2}))$$

So if the above is true the derivative can never be zero so the condition in the lemma holds. ■

Corollary 4.1. For a constellation to be optimal, the increase in the inter symbol distance as we move away from origin cannot be more than a certain limit, if $s_i > 0$ then $d_{i,i-1} < d_{i,i+1} < \sqrt{d_{i,i-1}^2 + 4\sigma^2 \ln(\frac{2i+1}{i+1})}$ and if $s_i < 0$ then $d_{i,i+1} < d_{i,i-1} < \sqrt{d_{i,i+1}^2 + 4\sigma^2 \ln(\frac{2i+1}{i+1})}$

Proof. Here also the idea is somewhat similar, first let us take $\{s_1, s_2, ..s_N\}$ to be optimal. Let $s_i > 0$, and here we change s_i and s_{i+1} only keeping the total power fixed, which would mean $s_i \delta s_i + s_{i+1} \delta s_{i+1} = 0$

$$\frac{\delta P_e(s_1, s_2, ..s_N)}{\delta s_i} = \tag{4.5}$$
$$Q'(\frac{d_{i,i-1}}{2\sigma}) - (1 + \frac{s_i}{s_{i+1}}) Q'(\frac{d_{i,i+1}}{2\sigma}) + \frac{s_i}{s_{i+1}} Q'(\frac{d_{i+1,i+2}}{2\sigma})$$

$d_{i,i+1} > \frac{s_i}{i}$ if $Q'(\frac{d_{i,i-1}}{2\sigma}) < (1 + \frac{s_i}{s_i + d_{i,i+1}}) Q'(\frac{d_{i,i+1}}{2\sigma})$ would mean the derivative in (4.5) can never be zero implying that such an arrangement cannot be optimal,

$$e^{-\frac{d_{i,i-1}^2}{4\sigma^2}} > (1 + \frac{s_i}{s_i + d_{i,i+1}}) e^{-\frac{d_{i,i+1}^2}{4\sigma^2}}$$
$$e^{-\frac{d_{i,i-1}^2}{4\sigma^2}} > (1 + \frac{1}{1 + \frac{d_{i,i+1}}{s_i}}) e^{-\frac{d_{i,i+1}^2}{4\sigma^2}}$$
$$e^{-\frac{d_{i,i-1}^2}{4\sigma^2}} > (\frac{2i+1}{i+1}) e^{-\frac{d_{i,i+1}^2}{4\sigma^2}}$$
$$-\frac{d_{i,i-1}^2}{4\sigma^2} > ln(\frac{2i+1}{i+1}) - \frac{d_{i,i+1}^2}{4\sigma^2}$$

$$P_{be}(s_1, s_2, ..s_N, c(1), ..c(N)) = \frac{1}{N\lceil \log_2 N \rceil} \sum_{i=1}^{N} \sum_{j=1}^{N} d(c(j), c(i)) P(s_j|s_i) \tag{4.6}$$

So, for the constellation to be optimal we have to have $d_{i,i-1} < d_{i,i+1} < \sqrt{d_{i,i-1}^2 + 4\sigma^2 \ln(\frac{2i+1}{i+1})}$

■

From this theorem we can also see that increment decreases with increasing SNR (decreasing σ) and for $\sigma \to 0$ we can see it approaches a uniform constellation.

4.2 Optimization in terms of Bit Error Rate

Here our aim is to minimize the BER for a PAM constellation given the power constraint but here as we would see that the objective function will depend on the coding scheme and it would be non convex unlike the SER case. But as we would see that the optimization problem can be converted into a convex problem under the condition that power constraint has to be more than a threshold. To develop this result we use the following results stated below. Let $c : [1, 2...N] \to \{0, 1\}^m$ be the coding function which maps the indices of the symbols to a string of length $m = \lceil \log_2 N \rceil$. Bit error rate function is given as (4.6)

Here $P(s_j|s_i)$ is the probability that symbol j is detected given s_i is sent and $d(c(i), c(j))$ is the hamming distance between the two codewords.

Lemma 4.3. For any N point PAM,

$$\frac{1}{\lceil \log_2 N \rceil} 2g(\frac{\sqrt{2P}}{2N\sigma}) \leq P_{be}(s_1, ..s_N) \leq 2g(\frac{\sqrt{P}}{2\sum_{i=1}^{\frac{N}{2}} i^2 \sigma}) \tag{4.7}$$

lower bound holds for all constellations and the upper bound is necessary for the constellation to be optimal in terms of BER

Proof. Here we want a good upper bound for the constellation to be optimal in terms of bit error rate and we know that closed form expression of uniform PAM given as, $P_{se}(s_1^u, ..s_N^u) = 2g(\frac{\sqrt{P}}{\sum_{i=1}^{\frac{N}{2}} i^2})$ $P_{se}(s_1, ..s_N) \leq 2g(\frac{\sqrt{P}}{2\sum_{i=1}^{\frac{N}{2}} i^2 \sigma})$.

From Lemma 3.3 $P_{be}(s_1,..s_N,c(1),..c(N)) \leq 2g(\frac{\sqrt{P}}{\sum_{i=1}^{2\frac{N}{2}} i^2\sigma})$

Now we establish the lower bound,

$$P_{se}(s_1,..s_N) = \frac{2}{N}\sum_{i=1}^{N} g(\frac{s_{i+1}-s_i}{2\sigma})\ ; G = \{s_1 < s_2.. < s_N\}$$

Here P_{se} is convex over set G and $\frac{2}{N}\sum_{i=1}^{N-1} g(\frac{s_{i+1}-s_i}{2\sigma}) \geq g(\sum_{i=1}^{N-1}\frac{s_{i+1}-s_i}{2\sigma}) = g(\frac{s_N-s_1}{N\sigma})$

Now we try to minimize

$$\begin{aligned} &\min g(\frac{s_N-s_1}{N\sigma}) \\ \text{or}\quad &\max(s_N - s_1) \\ &\text{subject to} \sum_{i=1}^{N} |s_i|^2 = P \end{aligned}$$

and a solution to this is obvious when $s_1^* = -s_N^* = \sqrt{\frac{P}{2}}$ and $g(2\frac{s_N-s_1}{2N\sigma}) \geq g(\frac{\sqrt{2P}}{N2\sigma})$ ■

Here our aim is to minimize $P_{be}(s_1,..s_N,c(1)..c(N))$ subject to the power constraint over all possible coding schemes. But since we know that gray coding is possible in a PAM constellation and gray coding is known to be optimal for uniform PAM.

Lemma 4.4. For any constellation to be optimal in terms of BER the separation between any two adjacent symbols

$d_{i,i+1} > g^{-1}(N\lceil\log_2 N\rceil g(\frac{\sqrt{P}}{2\sum_{i=1}^{\frac{N}{2}} i^2\sigma}))$

Proof. We know from Lemma 3.3 that $\frac{1}{\lceil\log_2 N\rceil}P_{se}(s_1,..s_N) < P_{be}(s_1,s_2,..s_N,c(1),..c(N)) < P_{se}(s_1,..s_N)$. This combined with Lemma 4.3 means that for a constellation to be optimal $\frac{1}{\log_2 N}P_{se}(s_1,..s_N) < 2g(\frac{\sqrt{P}}{2\sum_{i=1}^{\frac{N}{2}} i^2\sigma})$.

So now we have

$$\frac{1}{N}\sum_{i=1}^{N} g(\frac{s_{i+1}-s_i}{2\sigma}) < \lceil \log_2 N \rceil g(\frac{\sqrt{P}}{2\sum_{i=1}^{\frac{N}{2}} i^2\sigma})$$

$$\frac{1}{N} g(\frac{s_{k+1}-s_k}{2\sigma}) < \lceil \log_2 N \rceil g(\frac{\sqrt{P}}{2\sum_{i=1}^{\frac{N}{2}} i^2\sigma})$$

$$g(\frac{d_{i,i+1}}{2\sigma}) < (N\lceil \log_2 N \rceil g(\frac{\sqrt{P}}{2\sum_{i=1}^{\frac{N}{2}} i^2\sigma})))$$

$$d_{i,i+1} > g^{-1}(N\lceil \log_2 N \rceil g(\frac{\sqrt{P}}{2\sum_{i=1}^{\frac{N}{2}} i^2\sigma}))$$

■

This bound can be further tightened by using the exact BER of hierarchical gray coded PAM as the upper bound. So now we know that for any N PAM the distance between adjacent symbols has to be more than the threshold given in the previous theorem.

Corollary 4.2. For any constellation to be optimal in terms of BER in AWGN channel the separation between any two adjacent symbols

$d_{i,i+1} > 2\sigma Q^{-1}(N\lceil \log_2 N \rceil Q(\frac{\sqrt{P}}{2\sum_{i=1}^{\frac{N}{2}} i^2\sigma}))$

The above follows if we take the distribution as $f_G(g) = \delta(g-1)$

Observation 4.1. For any N PAM gray coded constellation the hamming distance $d(c(i), c(i+1)) = 1$ and $d(c(i), c(i+2)) = 2$

Since $d(c(i), c(i+2)) \leq d(c(i), c(i+1) + d(c(i+1), c(i+2)) = 2$ and the fact that $d(c(i), c(i+2) = 1$ is not possible (can be seen from construction) which means $d(c(i), c(i+2)) = 2$.

Theorem 4.2. In BER optimization of gray coded PAM constellation in AWGN channel global minimum is achieved beyond a average power threshold P^* within β distance $\beta << 1$

BER expression for any gray coded N PAM is given in (4.8) and we make use of this observation.

$$P_{be}(s_1,..s_N,c(1),..c(N)) = \frac{1}{N\lceil\log_2 N\rceil}\sum_{i=1}^{N}\sum_{j=1,j\neq i}^{N} d(c(j),c(i))\big(\big|g(\big|\frac{s_j+s_{j-1}}{2\sigma}-\frac{s_i}{\sigma}\big|)I(j>1)$$
$$-g(\big|\frac{s_{j+1}+s_j}{2\sigma}-\frac{s_i}{\sigma}\big|)I(j<N)\big|\big) = \frac{1}{N\lceil\log_2 N\rceil}\sum_{i=1}^{N}\big(g(\frac{d_{i,i+1}}{2\sigma})+g(\frac{d_{i,i-1}}{2\sigma})+g(\frac{d_{i,i+1}+\frac{d_{i+1,i+2}}{2}}{\sigma})$$
$$+g(\frac{d_{i,i-1}+\frac{d_{i-1,i-2}}{2}}{\sigma})\big)-2g(\frac{d_{i,i+1}+d_{i+1,i+2}+\frac{d_{i+2,i+3}}{2}}{\sigma})-2g(\frac{d_{i,i-1}+d_{i-1,i-2}+\frac{d_{i-2,i-3}}{2}}{\sigma})\big)+$$
$$\frac{1}{N\log_2 N}\sum_{i=1}^{N}\sum_{j\neq i,i-1,i+1,i-2,i+2} d(c(j),c(i))\big(\big|g(\big|\frac{s_j+s_{j-1}}{2\sigma}\frac{s_i}{\sigma}\big|)I(j>1)$$
$$-g(\big|\frac{s_{j+1}+s_j}{2\sigma}-\frac{s_i}{\sigma}\big|)I(j<N)\big|\big) \tag{4.8}$$

For AWGN channel we use $f_G(g)=\delta(g-1)$ and we see that in the sum of Q functions the dominant terms are coming from nearest neighbors and we use this fact in the following way.

$$\frac{Q(d_{i,i+1}+d_{i+1,i+2}+\frac{d_{i+2,i+3}}{2})}{Q(\frac{d_{i,i+1}}{2})} \leq \frac{e^{-\frac{\left(d_{i,i+1}+d_{i+1,i+2}+\frac{d_{i+2,i+3}}{2}\right)^2}{2}}}{e^{-\frac{d_{i,i+1}^2}{8}}} =$$
$$e^{\frac{-3d_{i,i+1}^2}{8}-\frac{d_{i+1,i+2}^2}{2}-\frac{d_{i+2,i+3}^2}{8}-d_i d_{i+1}-\frac{d_{i+1}d_{i+2}}{2}-\frac{d_{i+2}d_{i+3}}{2}}$$

This ratio combined with threshold $\alpha(P)$from lemma 4.4 gives that

$$\frac{Q(d_{i,i+1}+d_{i+1,i+2}+\frac{d_{i+2,i+3}}{2})}{Q(\frac{d_{i,i+1}}{2})} < e^{-3\alpha(P)^2}$$

So we see that there is an exponential decay which means that suppose $\beta << 1$ and $e^{-3\alpha(P)^2} < \beta$, for this to happen P has to be more than a certain value, calculated as $\alpha(P^*) = \sqrt{\frac{-1}{3}\ln(\beta)}$ This means that the bit error rate can be accurately

$$P_{be}(d_{1,2},d_{2,3},d_{3,4}) = \frac{1}{8}\big(Q(\frac{d_{1,2}}{2}) + Q(d_{1,2}+\frac{d_{2,3}}{2}) - Q(d_{1,2}+d_{2,3}+\frac{d_{3,4}}{2})\big) + \frac{1}{8}\big(Q(\frac{d_{3,4}}{2}) + Q(d_{3,4}+\frac{d_{2,3}}{2}) - Q(d_{3,4}+d_{2,3}+\frac{d_{1,2}}{2})\big) + \frac{1}{8}\big(Q(\frac{d_{1,2}}{2}) + Q(\frac{d_{2,3}}{2}) + Q(d_{2,3}+\frac{d_{3,4}}{2})\big) + \frac{1}{8}\big(Q(\frac{d_{3,4}}{2}) + Q(\frac{d_{2,3}}{2}) + Q(d_{2,3}+\frac{d_{1,2}}{2})\big)$$

---

approximated as,

$$P_{be}(s_1,..s_N,c(1),..c(N))' = \frac{1}{N\lceil\log_2 N\rceil}\sum_{i=1}^{N}\big(Q(d_{i,i+1})I(i<N-1) + Q(d_{i,i-1})I(i>1) + Q(d_{i,i+1}+\frac{d_{i+1,i+2}}{2})I(i<N-2) + Q(d_{i,i-1}+\frac{d_{i-1,i-2}}{2})I(i>2)\big)$$

Thus finally we have bit error rate as a sum of Q functions which makes it convex. This implies that this optimization problem can also be seen as a convex optimization problem which means the best possible constellation can be found.

In the case of gray coded 4-PAM exact BER is given by 4.9 The exact BER function is not convex but if we use the fact that $\frac{Q(d_{1,2}+d_{2,3}+\frac{d_{3,4}}{2})\big)}{Q(\frac{d_{1,2}}{2})} < \beta$ when $\alpha(P) > \sqrt{\frac{-1}{3}\ln(\beta)}$ In the SER case the objective function was a bit simpler but here as well we can use the same ideas to show that above a threshold of P the solution can be characterized in the same manner. Basically for optimal BER also the inter symbol distance increases as we move away from origin.

4.3 Optimization Results

Since we know that the problem is convex for SER in fading channel and for BER above a certain SNR in AWGN channel, in each case global minimum is attained. We see that optimal constellation changes with SNR.

In the plot below 4.2 we show the comparison between SER of 4 point uniform constellation PAM with different optimal constellation obtained at various SNR values. The improvement is not fixed (around 0.1 db between 1-4 db SNR) because

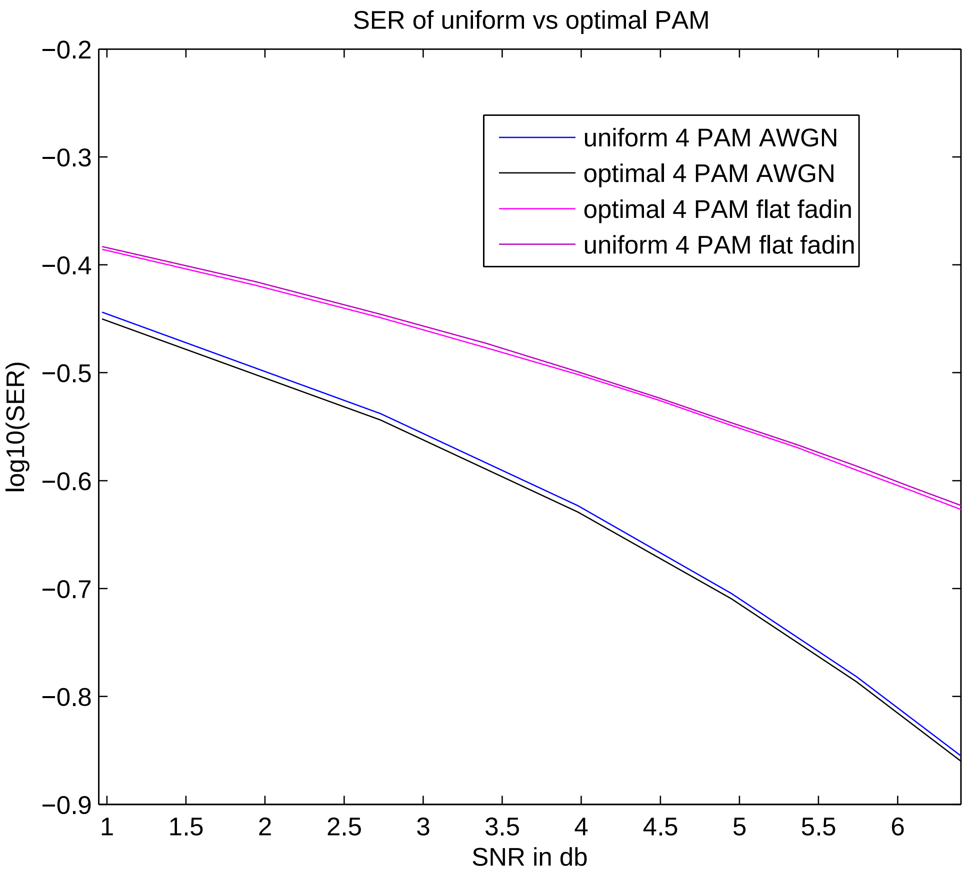

Figure 4.2: SER of uniform 4 PAM vs Optimal PAM

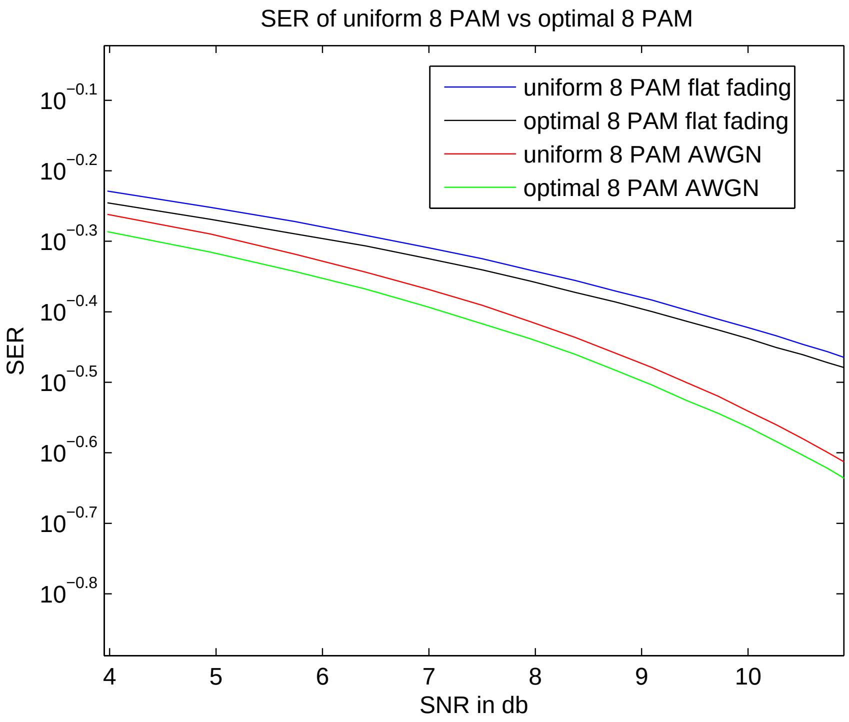

Figure 4.3: SER of uniform 8 PAM vs Optimal PAM

as SNR increases uniform comes closer to optimal as we have shown already.

Since the exact solution cannot be arrived at analytically what we do is from the geometry obtained we fit a polynomial in terms of parameter of the uniform constellation and obtain close to exact expressions as given in table on next page.

The improvement in SER and BER of binary reflected gray coded 8 PAM for AWGN and flat fading is shown below in figure 4.3 and 4.4. Improvement in the case of SER in AWGN channel is the maximum around 0.25db. We can also arrive at a similar table which gives an accurate expression for symbol points in the optimal constellation.

Gray coding of 8 PAM is given as $\{c(1), ..c(8)\}$ is $\{000, 001, 011, 010, 110, , 111, 101, 100\}$

4.4 Conclusion

In this chapter we come up with optimization formulation for 1-D constellations in both AWGN and fading channels. For SER minimization the problem is convex in both AWGN and fading channel and for BER minimization the problem is shown

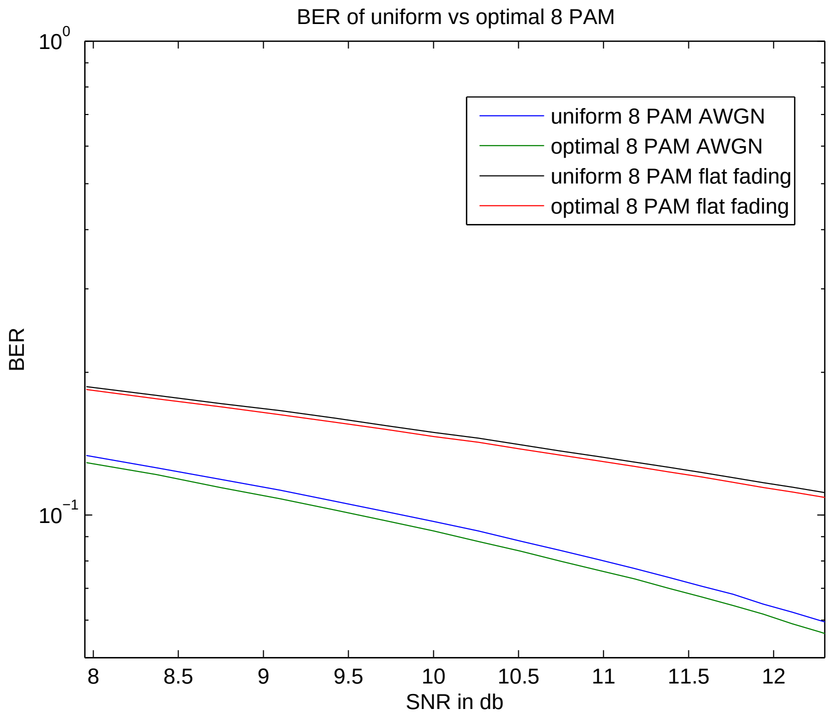

Figure 4.4: BER of gray coded uniform 8 PAM vs Optimal PAM

	Uniform 4 PAM	Optimal 4 PAM AWGN	Optimal 4 PAM flat fading
s_1	$\frac{-3d}{2}$	$\frac{-3d}{2}+\frac{0.112}{d}+\frac{0.0193}{d^2}-\frac{0.157}{d^3}-\frac{0.0857}{d^4}$	$\frac{-3d}{2}+\frac{0.4135}{d}-\frac{1.35}{d^2}+\frac{1.8991}{d^3}-\frac{0.9562}{d^4}$
s_2	$\frac{-d}{2}$	$\frac{-d}{2}-\frac{0.347}{d}+\frac{0.0067}{d^2}+\frac{0.1886}{d^3}-\frac{0.067}{d^4}$	$\frac{-d}{2}-\frac{1.28}{d}+\frac{4.19}{d^2}-\frac{5.93}{d^3}+\frac{2.99}{d^4}$
s_3	$\frac{d}{2}$	$\frac{d}{2}+\frac{0.347}{d}-\frac{0.0067}{d^2}-\frac{0.1886}{d^3}+\frac{0.067}{d^4}$	$\frac{d}{2}+\frac{1.28}{d}-\frac{4.19}{d^2}+\frac{5.93}{d^3}-\frac{2.99}{d^4}$
s_4	$\frac{3d}{2}$	$\frac{3d}{2}-\frac{0.112}{d}-\frac{0.0193}{d^2}+\frac{0.157}{d^3}+\frac{0.0857}{d^4}$	$\frac{3d}{2}-\frac{0.4135}{d}+\frac{1.35}{d^2}-\frac{1.8991}{d^3}+\frac{0.9562}{d^4}$

to be convex beyond a certain SNR threshold, in AWGN channel, implying global minimum is achieved. In the next chapter we aim to solve the same problem in the context of 2-D constellations.

Chapter 5

Optimizing 2-D Constellations in terms of Error Rate at Finite SNR

In this chapter we deal with the optimization problem for the case of 2-Dimensional constellations at any finite SNR. We formulate the problem in a numerical optimization framework and using standard optimization procedures we show the optimum constellations for the case of 8 and 16 point constellations in AWGN channnel.

In the first section we show numerical evaluation of SER/BER in both AWGN and fading channel can be useful in getting the optimal solution up to a tolerance level. Upon formulating the numerical optimization problem in the next section we use interior point methods to come up with optimal solutions for 8 and 16 point constellations. In the last section we conclude the chapter and lay the theme for next chapter.

5.1 Numerical Optimization to Minimize SER/BER

As we can see that the objective function which is symbol error rate is not convex and there is no closed form to it for any general arrangement of points. At best the SER/BER expressions can be written in terms of certain gaussian 1/2-dimensional integrals. Since we need to come up with optimal geometries which give the least SER/BER for any N point constellation at a fixed SNR, the best way to go about is numerical optimization. Consider a set $G = \{(s_1, s_2 .... s_N) \sum_{i=1}^{N} |s_i|^2 \leq P\}$ and let $\sigma^2 = 1$ in a general fading channel.

$$
\begin{aligned}
&\alpha(x,y) = \arg \min_{i=1,..N} |(x,y) - s_i|_2 \\
&P_{se}(s_1, s_2, ..s_N) = 1 - \frac{1}{N} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{2\pi} e^{-(\min_{i=1,...N} \|(x,y)-\sqrt{g}s_i\|_2^2)} f_G(g) dx dy dg \\
&P_{be}(s_1, s_2, ..s_N, c(1), .c(N)) \\
&= \frac{1}{N\lceil \log_2 N \rceil} \sum_{i=1}^{N} \int_0^\infty \int_{-\infty}^\infty \int_{-\infty}^\infty d(c(\alpha(x,y), i) e^{-(\|(x,y)-\sqrt{g}s_i\|_2^2)} f_G(g) dx dy dg
\end{aligned}
$$

As already said above we would be considering the numerical evaluation of the SER and BER expressions above. To do that we bring the notion of tolerance which basically is the limit for the error in the numerical evaluation from the actual exact value, in our case it is ϵ. For numerical evaluation we can say that for any fading distribution there would be a value g^* , $P(G > g^*) < \delta$ where δ is close to zero.

We consider a square grid of discrete points shown in the figure which goes from $-r\sqrt{g^* P} : r\sqrt{g^* P}$ where $r > 1$, gain is also discretized into K intervals. On this discretized domain, we use trapezoidal integration over this region to come up with expression P_e^n within tolerance level of ϵ as follows

$$
\begin{aligned}
&P_{se}^n(s_1, s_2, ...s_N) = 1 - \frac{1}{N} \frac{1}{2\pi} \sum_{t=1}^{K} \sum_{i=1}^{M} \sum_{j=1}^{M} e^{-(\min_{k=1,...N} \|(x_i,y_j)-\sqrt{g_t}s_k\|_2^2)} \Delta x \Delta y \Delta g \\
&\Delta x = \Delta y = 2\frac{r\sqrt{g^* P}}{M} \quad \Delta g = \frac{g^*}{K}
\end{aligned}
$$

We want to choose Δx , Δg such that $|P_{se}(s_1, ..s_N) - P_{se}^n(s_1, ..s_N)| < \epsilon$ and the error in the above comes due to two factors, one due to not considering the integral over the complete space and second due to trapezoidal approximation. We select the domain such that we can reduce the error to below the tolerance value. Here we show the analysis for the case of AWGN channel where the square grid is $-r\sqrt{P} : r\sqrt{P}$

given in figure 5.1. The error due to the first factor can be seen as

$$e_1(s_1,..s_N) = \int_{-\infty}^{\infty}\int_{r\sqrt{P}}^{\infty} e^{-(\min_{i=1,...N}|(x,y)-s_i|_2^2)}dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{-r\sqrt{P}} e^{-(\min_{i=1,...N}|(x,y)-s_i|_2^2)}dxdy + \int_{-r\sqrt{P}}^{r\sqrt{P}}\int_{-\infty}^{-r\sqrt{P}} e^{-(\min_{i=1,...N}|(x,y)-s_i|_2^2)}dxdy + \int_{-r\sqrt{P}}^{r\sqrt{P}}\int_{r\sqrt{P}}^{\infty} e^{-(\min_{i=1,...N}|(x,y)-s_i|_2^2)}dxdy$$

The above error expression can be bounded above using the following idea, basically the minimum distance of a point from the boundary of the grid is $(r-1)\sqrt{P}$ assuming the symbol has maximum possible power P.

$$e_1(s_1,..s_N) < \int_{\theta=0}^{2\pi}\int_{(r-1)\sqrt{P}}^{\infty} \frac{1}{2\pi} re^{-r^2} d\theta dr = e^{-(r-1)^2 P}$$

Error due to trapezoidal integration can be calculated using the following result [19], $|\int_a^b f(x)dx - \frac{(b-a)}{M}(\frac{f(a)+f(b)}{2} + \sum_{i=1}^{M} f(a+i\frac{b-a}{M})| < \frac{(b-a)^3}{12M^2}$

From here we can derive the 2-D formula and from that we would have the bound on error of second type

$$e_2(s_1,..s_N) < \frac{4P^2}{M^2}$$

In fact the $e_2(s_1,..s_N)$ would be bounded by $\frac{4P^2}{M^4}$ as P becomes large (proof of this is not given here). So what we can see is that we can by appropriately choosing r and Δx come up with P_{se}^n which is within ϵ tolerance level of the actual value.

Suppose $P_{se}(s_1^*,..s_N^*)$ be the minimum symbol error rate that is achieved by any constellation in G and the optimization of P_{se}^n would give an optimal solution within the tolerance level. So we have

$$\begin{aligned} &\min P_{se}^n(s_1,..s_N) \\ &\text{subject to} \sum_{i=1}^{N} |s_i|^2 = c \end{aligned}$$

In the above optimization problem we can see that the objective function is a non convex one and our aim is to attain the global minima. So to do that we need to search all the local minimum values. Basically on the level set of constraint function which would be a $2N$ dimensional sphere we need to find all such minimum values. It could be that there are infinite such minima but interestingly there would be only finitely many distinct local minimum values, now we would establish this claim.

First of all we can see that $P^n_{se}(s_1, ..s_N)$ can be written in the form of a superposition of integrals of smooth functions. From this we can see that for any small change in $\Delta P^n_{se} < \delta$we would have a corresponding norm ball s'_i s.t $|s'_i - s_i|_2 < \epsilon$. Also we can see from simple ideas that the derivative of the P_{se} would also be continuous.

Now assume that there are infinitely many distinct local minimum values for the optimization problem above. For any $\beta > 0$ we would be able to find local minima separated by a distance lesser than β. These local minima have must same value of function since having distinct values at arbitrary small separation is in straight violation with continuity.

Observation 5.1. There are only finitely many distinct local minimum values of the optimization problem given above.

To arrive at each of these local minimum values we use interior point methods from different start points. Using a large number of random start points we arrive at different local minimum values and with a high probability we can say that the global minimum value is attained.

Observation 5.2. Optimal constellation obtained is SNR dependent, basically the geometry or relative arrangement of points changes with SNR.

On the same lines as symbol error rate, we can show that bit error rate can be evaluated numerically. For the case of SER in AWGN channel , the error term for integral outside the square grid goes as $e^{-(r-1)^2 P}$. The same bound can be used for bit error rate from lemma 3.3 and the error due to trapezoidal integration goes as $\mathcal{O}(\frac{1}{M^2})$.

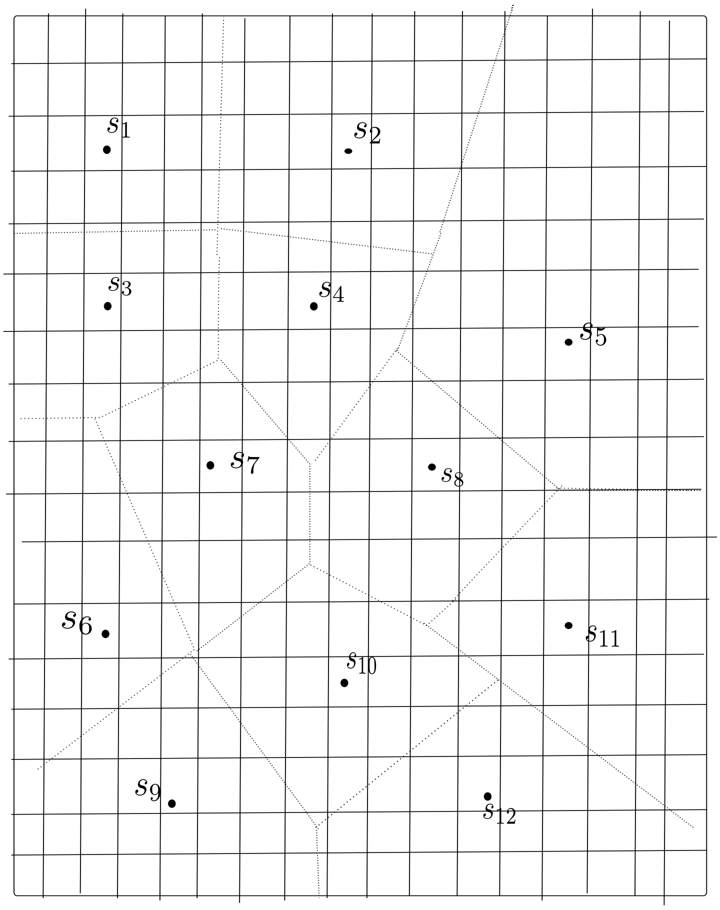

Figure 5.1: Grid Structure for evaluating SER of any N point constellation

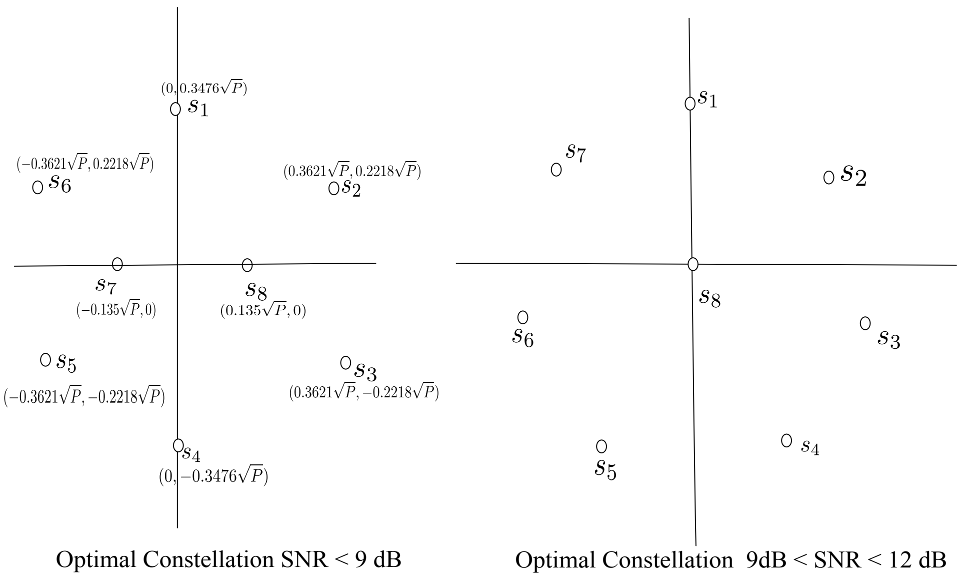

Figure 5.2: 8 point Optimal constellations obtained at different SNR values

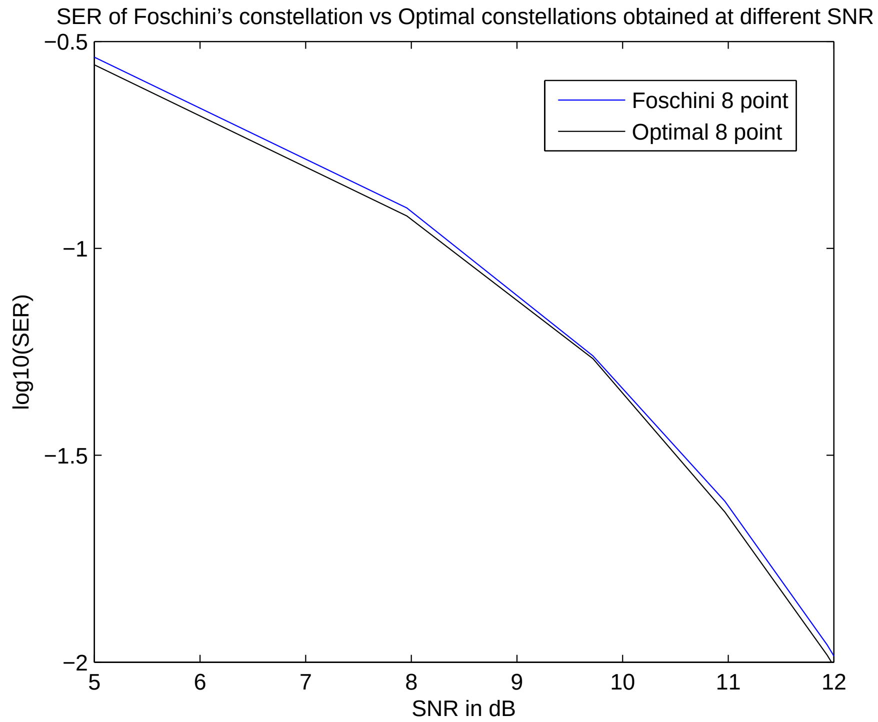

Figure 5.3: log(SER) vs SNR of optimal 8 point vs foschini's constellation

5.2 Optimization Results for 8 and 16 point Constellations

Interior point methods [20]are used for numerical optimization of 8 and 16 point SER expressions at different SNR values. Using these procedures from multiple random feasible start points helps us compare the local minima and arrive at global minimum with a high probability. In order to do this efficiently and arrive at global minima with a high probability global search procedures are used in matlab with interior point method as the local solver. For the case of 8 points optimization is done at different SNR values and the SER of the optimal constellation is shown compared with asymptotically optimal Foschini's constellation. Shape of the optimal constellation in terms of SER changes, it is of type-1 as shown in figure 5.2 before 9db and it changes to 1-7 uniform constellation from 9-12 db. The improvement obtained in comparison to Foschini's constellation is plotted in figure 5.3.

Same global optimization from multi start points is performed for 16 point con-

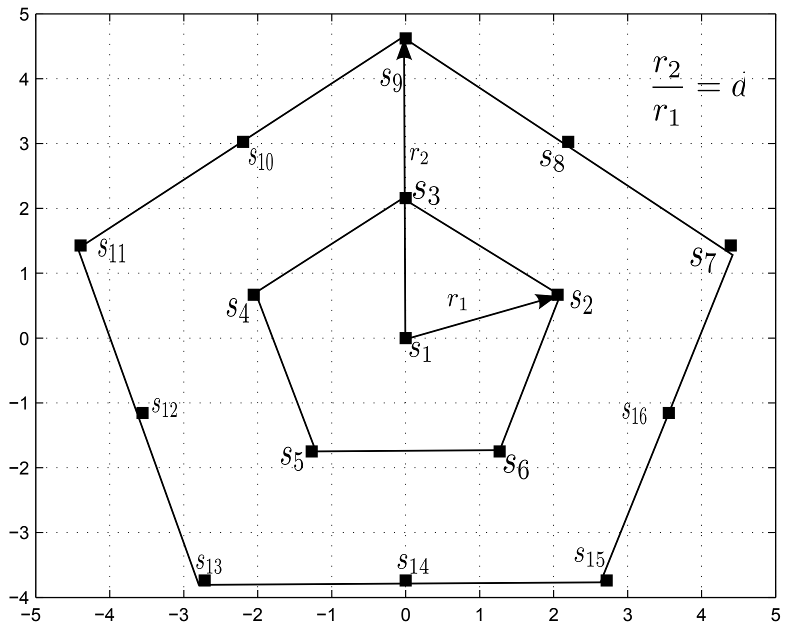

Figure 5.4: Pentagonal constellation optimal between SNR 11 to 17dB

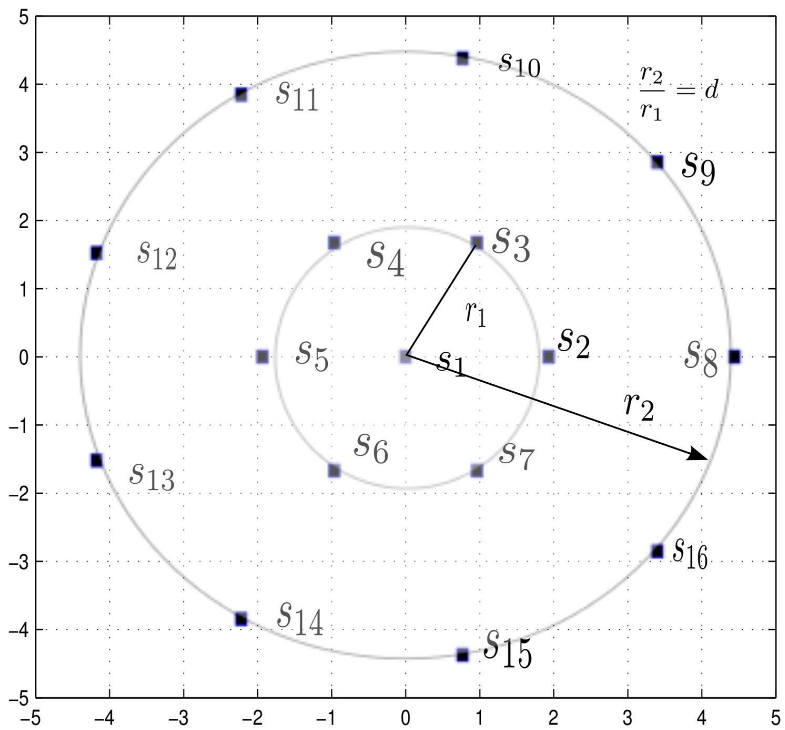

Figure 5.5: 1-6-9 uniform constellation optimal between 5-11 dB

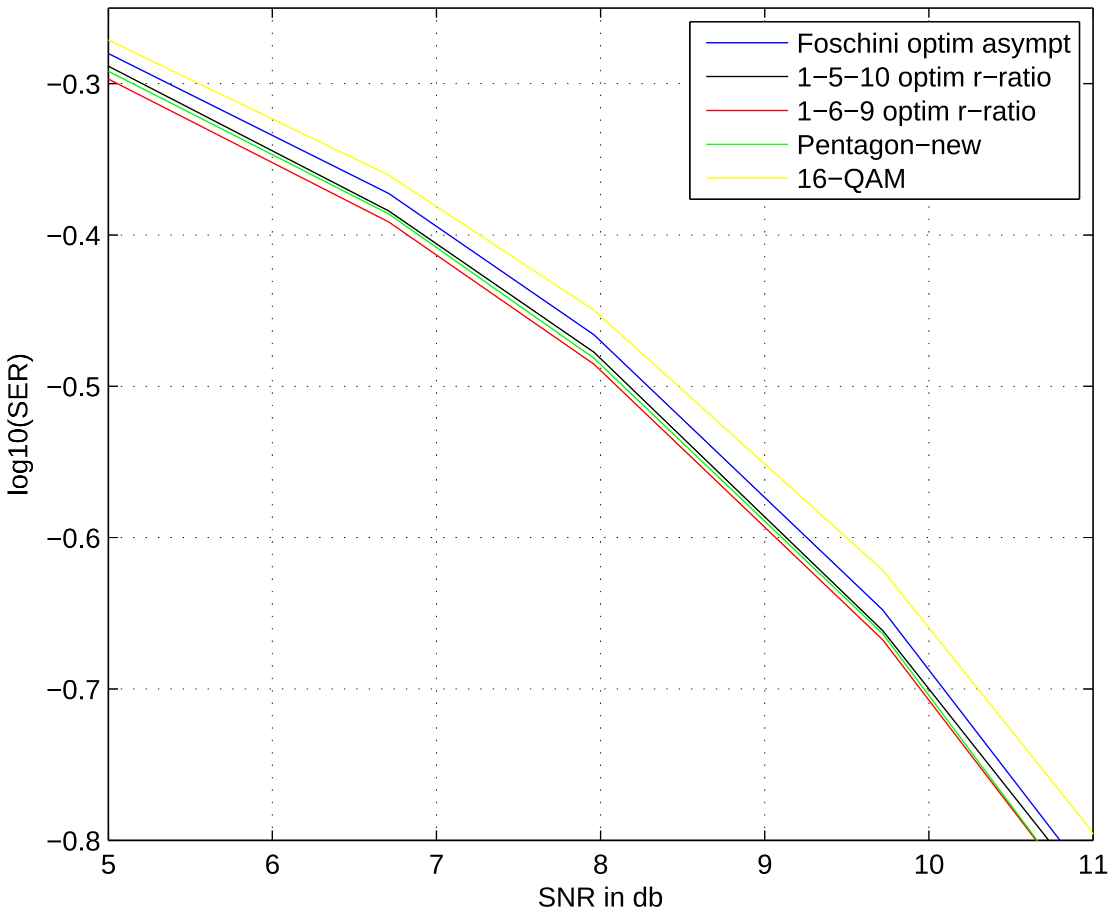

Figure 5.6: log(SER) vs SNR of optimal 16 point vs other constellations

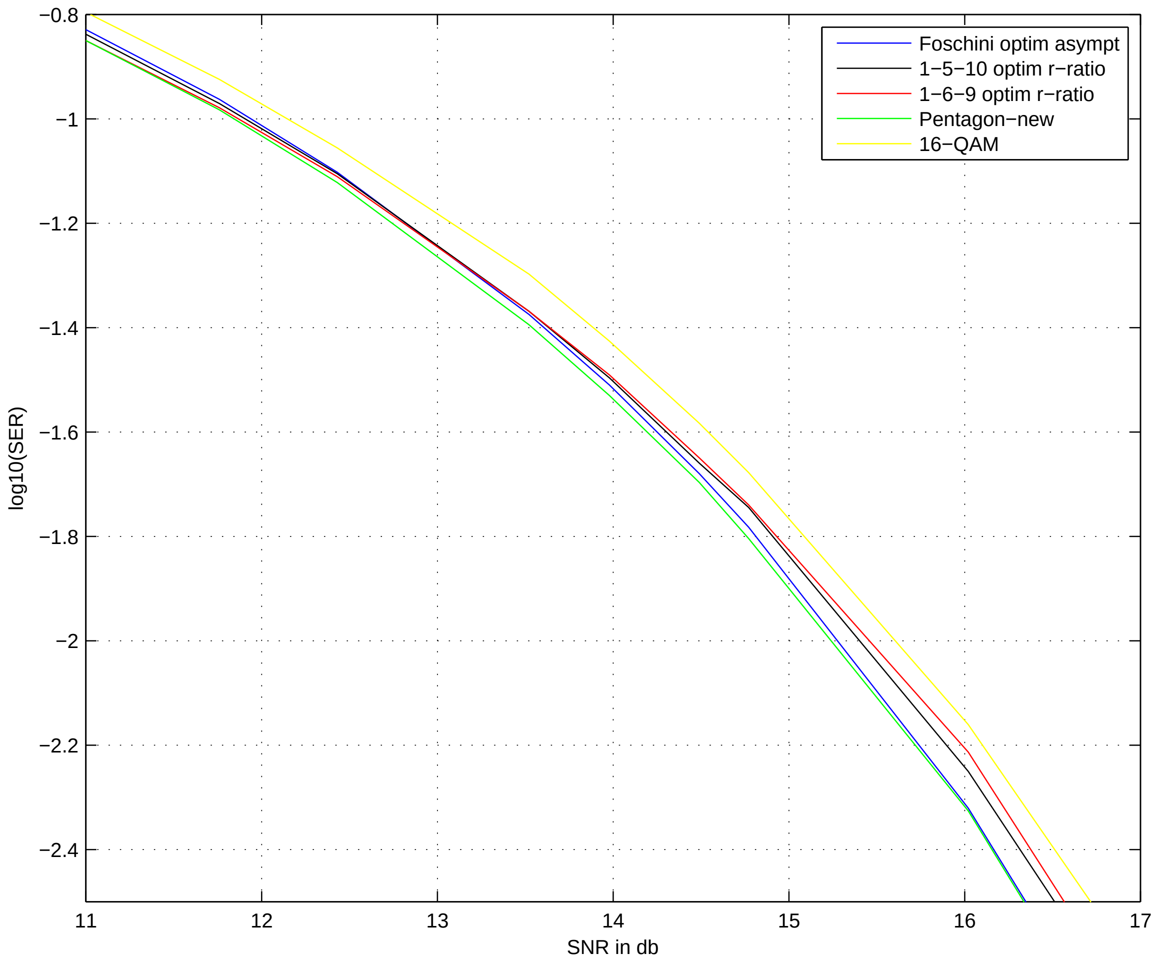

Figure 5.7: log(SER) vs SNR of optimal 16 point vs other constellations

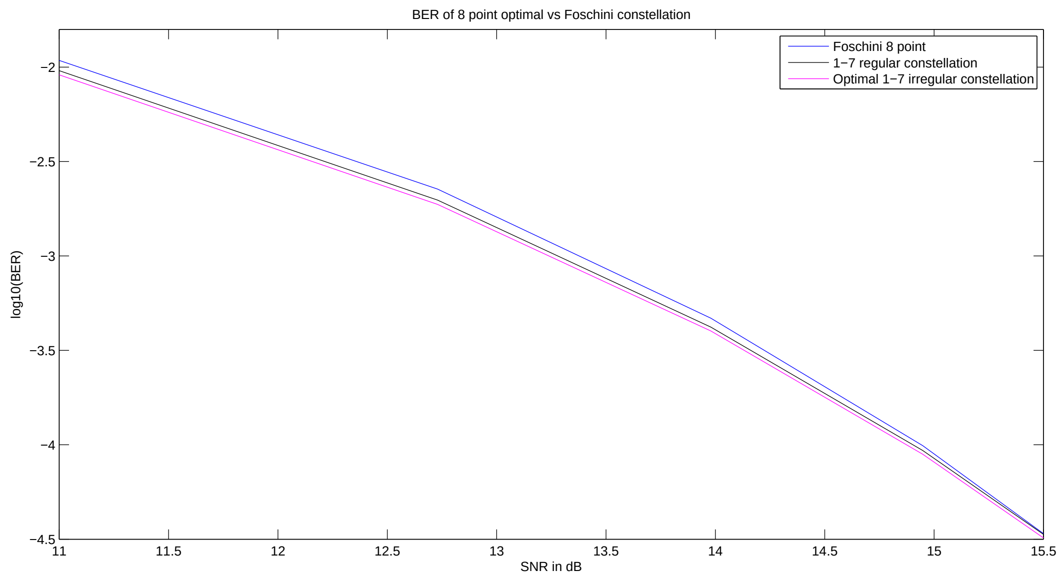

Figure 5.8: log(BER) vs SNR of optimal 8 point vs Foschini's constellation

stellation as well for different SNRs and the results are given below. 1-6-9 Constellation shown in figure 5.5 is optimal at lower SNRs (5-11 dB). But it is important to note that the ring ratio of optimal 1-6-9 constellation depends on the SNR value. The optimal ring ratio decreases from 2.85 at 5 dB to 2.09 at 10dB. A new type of constellation, pentagonal, figure 5.4, is optimal constellation from SNR range 11-17 dB. Here also ring ratio of the optimal constellation changes with SNR from 2.40 at 11 dB to 2.1 at 15 dB. Performance of the optimal constellation and the other constellations in terms of SER is shown in figure 5.6 and 5.7.

Now we see the BER optimization of 8 point constellation. Before we do this we should keep one thing in mind optimal constellation in terms of BER would depend on the coding scheme being used. In general gray coding is the best possible scheme but most of the times gray coding is not possible. As we see that in the case of 8 PSK gray coding is possible but PSK is not an optimal arrangement because of of the less nearest neighbor distance. Different labeling schemes were used and optimal solution was obtained and it was found that minimum bit error rate is achieved for the irregular constellation shown in 5.9 with labeling scheme same as 1-7 regular constellation. The improvement from foschini's constellation is shown in figure 5.8.

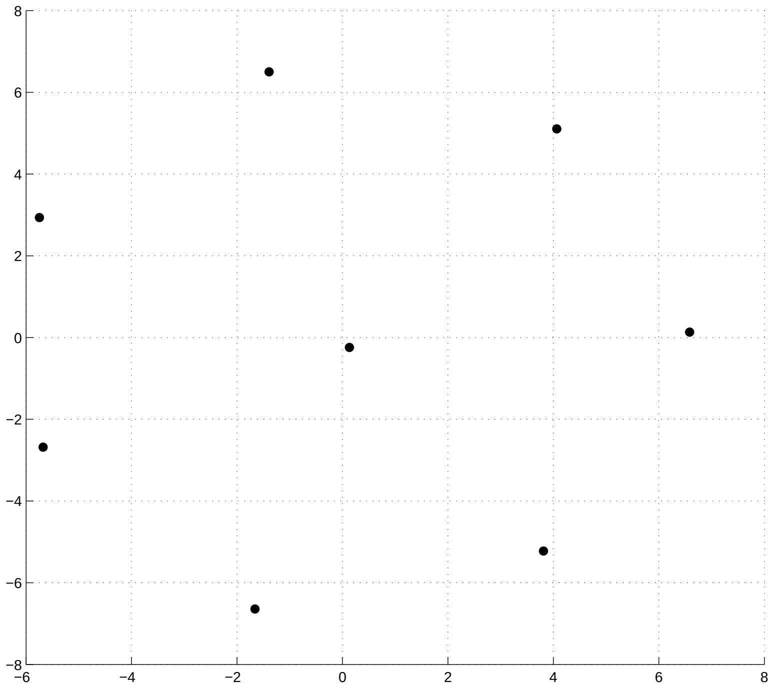

Figure 5.9: 8 point optimal irregular constellation

Interestingly the optimal constellation obtained is not regular, the distance between the symbol at the center and other symbols is not the same, reason being that it is not at the same hamming distance from all the points.

For the case of 16 point constellations in terms of bit error rate different labeling schemes were used to obtain optimal solutions. Gray coded uniform square QAM 5.11 where $d_1 = d_2$ is the best known constellation in low SNR range. Using optimization procedures, we were able to obtain gray coded non uniform QAM $d_2 > d_1$ as the optimal constellation, improvements are shown in figure 5.10

5.3 Conclusion

In this chapter we saw numerical formulation of the 2-D constellation optimization in terms of both SER and BER in AWGN and fading channel. We came up with optimal solutions in terms of both SER and BER in AWGN channel for 8 and 16 point case. Up till now we have been able to arrive at the optimal constellations for

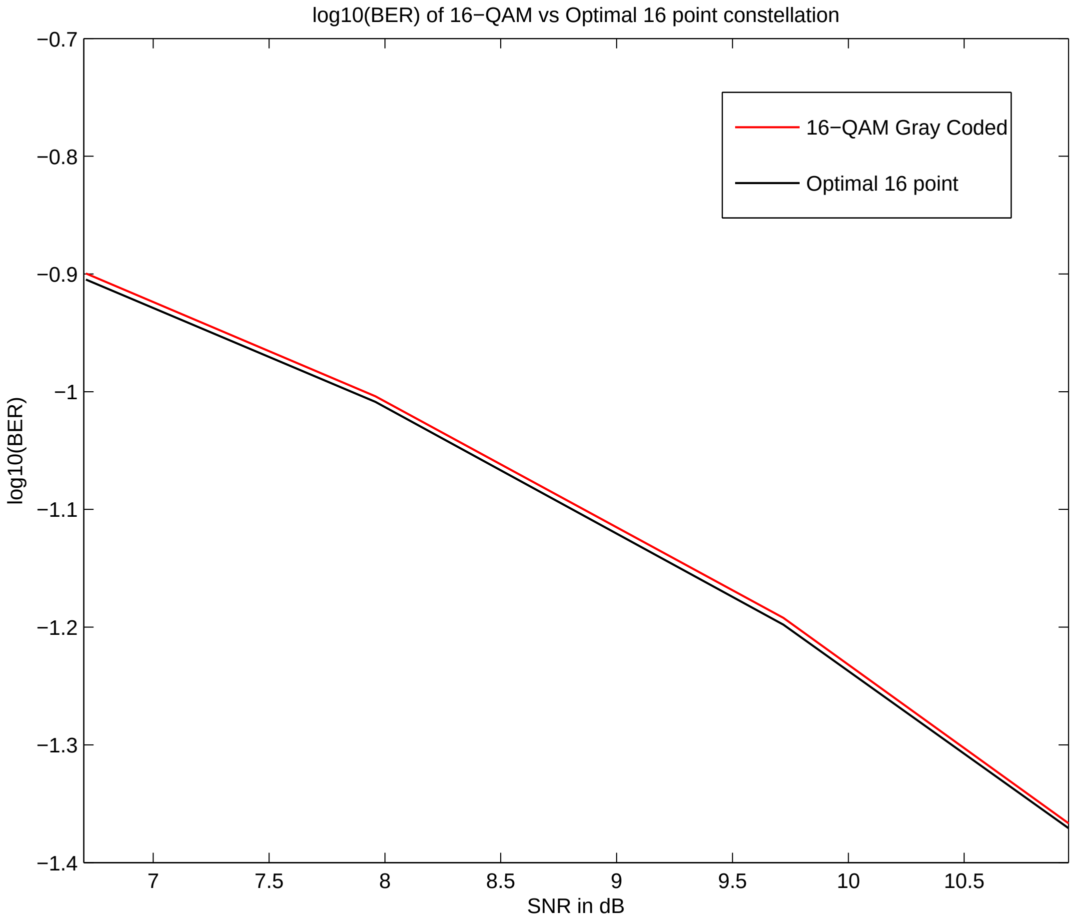

Figure 5.10: 16 point optimal non uniform QAM vs uniform QAM constellation

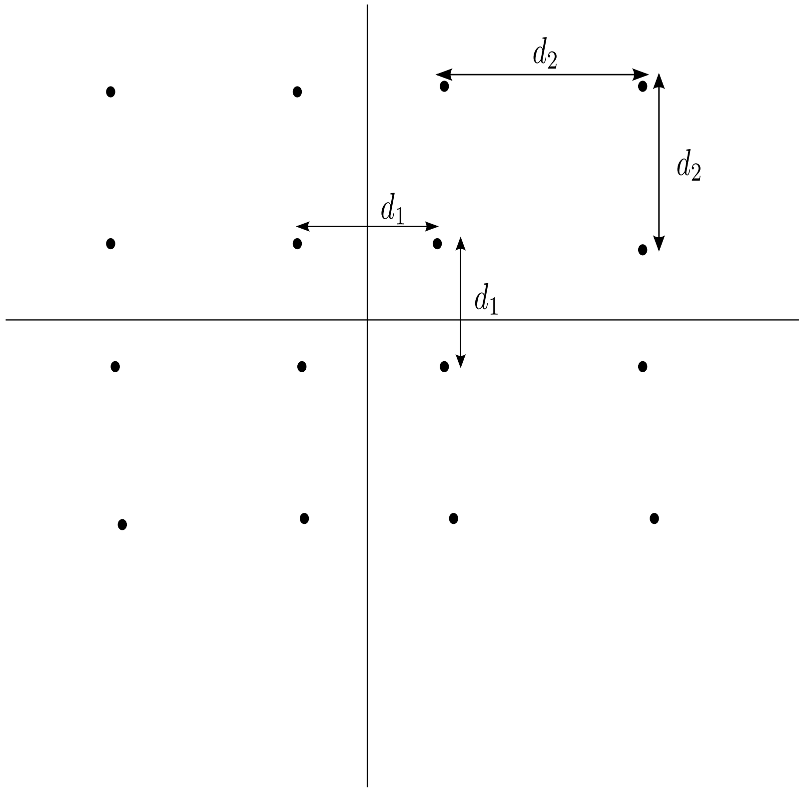

Figure 5.11: 16 point optimal non uniform QAM constellation

both 1-D and 2-D constellations for a given power constraint, in the next chapter we analyze an interesting application of the solutions obtained till now.

Chapter 6

Adaptive Transmit Power and Constellation Allocation in Fading Channel

In this chapter we show that if the constellation and the transmit power are assigned based on channel gain keeping the average power constraint satisfied improvement in error rate is possible. We formulate the same as an optimization problem and show the semi analytic solution for optimal power allocation in terms of optimal error rate as a function of power.

In the first section we formulate the case when the transmitter is allowed to vary the transmit power and the constellation based on fading gain as an optimization problem in terms of SER/BER. In the next section we show that for minimizing SER of 1-D constellation the problem is convex followed by the section on optimization results where we illustrate the improvement possible with the help of an example.

6.1 Adaptation of Transmit Power with Channel Gain

Up till now what we have done is given a fixed transmit power we find a 1-D/2-D constellation which gives the minimum SER/BER for the given nature of channel. Constellation obtained depends on the power, P, $(s_1^*(P), s_2^*(P), ..s_N^*(P))$ and the minimum SER is given as $P_e^*(P) = P_e(s_1^*(P), ..s_N^*(P))$. Here in this chapter we illustrate our idea only on optimization in terms of symbol error rate.

In a fading channel when the transmitter knows the gain $G = g$, whose pdf is given as $f_G(g)$ and based on this the transmitter should decide what constellation to use for transmission so that the average SER is minimized. This can be formulated as the following optimization problem

$$\min \int_0^\infty P_e(s_1(P_t(g)), ...s_N(P_t(g)))f_G(g)dg$$
$$\text{subject to} \int_0^\infty P_t(g)f_G(g)dg = \bar{P_T}$$
$$\sum_{i=1}^{N} |s_i(P_t(g))|^2 = P_t(g)$$

We can simplify the above by breaking the problem into two parts, for a particular value of gain g, $P_t(g)$ is the transmit power and for this particular transmit power level we know the optimal constellation and $P_e^*(P_t(g))$. Here we would like to comment that this optimal closed form expression is gettable for 1-D case but for 2-D case the expression is complicated as it would depend on which shape of constellation is being used in that particular SNR range. Now the optimization problem can be restated as

$$\min \int_0^\infty P_e^*(P_t(g))f_G(g)dg$$
$$\text{subject to} \int_0^\infty P_t(g)f_G(g)dg = \bar{P_T}$$

The lagrangian for the above optimization problem is defined below,

$$\mathcal{L}(P_t(g), \lambda) = \int_0^\infty P_e^*(P_t(g)g) f_G(g) dg + \lambda(\int_0^\infty P_t(g) f_G(g) dg - \bar{P}_T)$$

The optimal power allocation scheme has to satisfy

$$\begin{aligned}
&\frac{\partial \mathcal{L}(P_t(g), \lambda)}{\partial P_t(g)} = 0 \\
&g P_e^{*'}(P_t(g)g) f_G(g) + \lambda f_G(g) = 0 \\
&P_t(g) = \frac{P_e^{*'-1}(\frac{-\lambda}{g})}{g} \\
&\int_0^\infty \frac{P_e^{*'-1}(\frac{-\lambda}{g})}{g} f_G(g) dg = \bar{P}_T
\end{aligned}$$

Here $P_e^{*'}$ is the derivative of P_e^* , $P_e^{*'-1}$ is the inverse of the derivative assuming it exists.

6.2 Convex Formulation for Symbol Error Rate of 1-D constellation

The above minimization problem can be shown to be convex for the case of symbol error rate minimization of 1-D constellation. $\{s_1(g), ..s_N(g)\}$ denote the set of constellations which depend on the channel gain and they can also be denoted by $\{s_1(g), d_1(g), .d_{N-1}(g)\}$ where d_i is the inter symbol distance. Here the idea used is similar to 4.3 and is given as follows,

$$\begin{aligned}
&\min \int_0^\infty \frac{2}{N} \sum_{i=1}^{N} Q(\sqrt{g} d_i(g)) f_G(g) dg \\
&\text{subject to } \sum_{k=1}^{N} |s_1(g) + \sum_{i=1}^{k-1} d_i(g)|^2 f_G(g) dg \leq \bar{P}_T \\
&d_i(g) \geq 0 \; \forall i, \; g > 0
\end{aligned}$$

By discretizing the channel gain one can gain insight into the fact that the above problem is convex. Once we discretize, the integral in objective and constraint in the above formulation is replaced by summation of convex functions.

6.3 Optimization Results

In the figure 6.1 we show the solution for optimal power allocation of 8 point 1-D constellation vs channel gain. In this case we have the average transmit SNR , 11 dB and the optimal power and constellation are used depending upon the channel gain. If we do not adapt the transmit power and use uniform PAM SER is 0.340 and if we do not adapt and use non uniform optimal PAM then SER is 0.330 and by adapting it reduces to 0.316, the gains become larger as number of points in the constellation become large (here only 8).

6.4 Conclusion

In this chapter we came up with solution to the optimization problem where the transmitter is allowed to adapt the optimal constellation based on channel gain keeping the size and dimensionality of constellation fixed. Convex formulation for the case of SER minimization in 1-D showed global minimum can be achieved. Next we go on to the chapter where we conclude our work and throw some light on future work that is possible.

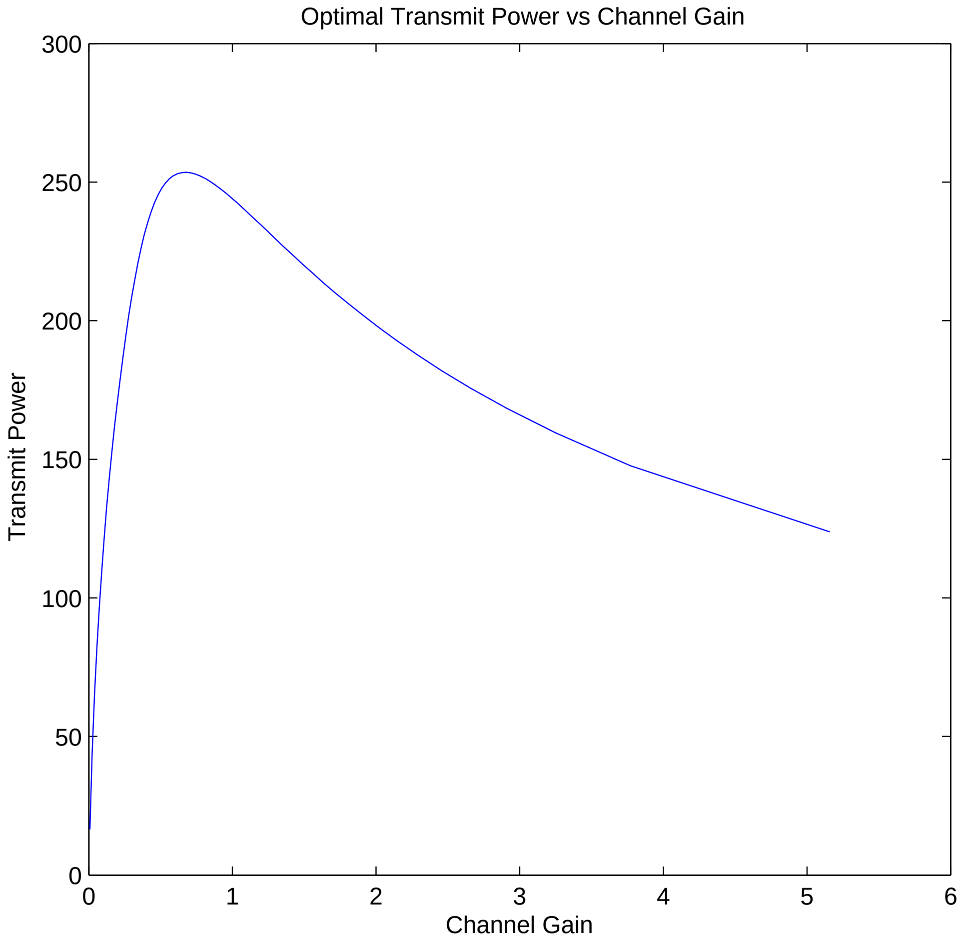

Figure 6.1: Adaptive Power Waterfilling for 8 point 1-D case

Chapter 7

Conclusions and Future Work

7.1 Conclusions

In this work we have been able to close some open ends in the understanding of best signal geometries. First of all for the case of 1-D constellations we have been able to show the best signal constellations which achieve global minimum with respect to symbol error rate in any given fading channel. Next obvious problem which is optimization of bit error rate of 1-D constellations in both AWGN and fading channel is analyzed and close to globally optimum solutions are arrived at. For the case of 2-D constellations we establish that at asymptotes optimizing symbol and bit error rate is equivalent in both AWGN and fading channels and the optimal constellations in both these cases form a lattice of equilateral triangles. Also the necessary conditions given by Fochini et.al for asymptotic optimality in AWGN channel are shown to be inaccurate and we propose alternate necessary conditions. For the case of finite SNR in AWGN channel, we come up with optimal 8 and 16 point constellations in terms of both SER and BER. At the end we show further improvements in the case of fading channel are possible for both 1-D and 2-D constellations if the transmitter optimally adapts to the channel gain and allocates the transmit power and constellation.

7.2 Future Work

Some interesting directions for future work are

- To obtain optimal 2-D constellations of large size at any fixed SNR, numerical evaluation of symbol/bit error rate needs to be efficient to obtain close to optimal solutions. Efficient ways to compute the symbol and bit error rate of any constellation are there in literature which can be used to get close to best solutions efficiently.
- The above work can be extended to multi dimensional constellations for both symbol and bit error rate. Also the necessary conditions arrived at for asymptotic optimality can be extended to the case of multi dimensional constellations
- In our work we have assumed the data rate to be constant which can be relaxed to obtain adaptive schemes. If the transmitter's objective is to minimize the error rate keeping the average power constraint satisfied and also keeping the data rate above a threshold, optimal adaptive schemes based on our work can be developed. Although adaptive schemes exist in literature but they are suboptimal since they allow one to choose constellations only from fixed family like QAM or PSK.

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